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STATISTICAL PROPERTIES OF NON-LINEAR FORCES OF SEA WAVES ON A VERTICAL WALL

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1 ABSTRACT

The statistical properties of some narrow-band second-order processes in the mechanics of sea waves in front of a vertical wall are investigated. Arena & Fedele (2002) have shown that the asymmetry between the crest and the trough distributions on a vertical wall is markedly greater than the asymmetry crest-trough in an undisturbed field, either for the free surface displacement or for the fluctuating wave pressure at a fixed depth. In this paper the narrow-band second-order wave force and overturning moment on the vertical wall are derived by analytical integration of the wave pressure. It is obtained that for a fixed threshold of the probability of exceedance, if kd is smaller than 1.38 *(being* k *the wave number and* d *the bottom depth), positive peaks of the wave force process (that occur with the wave crests of free surface displacement on the wall) are greater than absolute value of negative peaks (that occur with the wave trough).*
The wave force is a quasi-symmetric process for kd=1.38*: the non-linearities are weak and both the positive peak distribution and the negative peak (in absolute value) distribution are given by the Rayleigh law with good approximation.*
Finally it is noteworthy that for kd>1.38 *we find that the wave troughs of surface displacement at wall produce negative peaks of the wave force that are (in absolute value) greater than positive peaks produced by wave crests. The predictions are in good agreement with the data of a small-scale field experiment (Arena, 1995; Boccotti, 2000).*

2 INTRODUCTION

According to the linear theory of wind-generated waves (Longuet-Higgins, 1963; Phillips, 1967) both the free surface displacement and the fluctuating wave pressure in front of a vertical wall represent a random Gaussian process of time. As consequence the wave crest and wave trough have a Rayleigh distribution for an infinitely narrow spectrum (Longuet-Higgins, 1952), either for the linear free surface displacement or for the linear fluctuating wave pressure.

In this paper the statistical properties of some narrow-band second-order processes in the mechanics of sea waves in front of a vertical wall are investigated. For this purpose we consider a family ψ of stochastic processes, which both the second-order free surface displacement and fluctuating wave pressure on a vertical wall belong to. The probability density function and the probabilities of exceedance of both the absolute maximum and the absolute minimum of this stochastic family were obtained by the authors (Arena & Fedele, 2002). It is shown that some non-linear effects for sea waves on a vertical wall are greater than the non-linear effects in an undisturbed wave field: the asymmetry between the crest and the trough distributions on a vertical wall is markedly greater than the asymmetry crest-trough in an undisturbed field, either for the free surface displacement or for the fluctuating wave pressure at a fixed depth. These results well agree with the data of a small-scale field experiment (Boccotti, 2000).

Finally the second-order random wave force F and overturning moment M on the vertical wall are derived by analytical integration of the fluctuating wave pressure. It is obtained that both these two processes, for an infinitely narrow spectrum, belong to the stochastic family ψ too. The statistical properties of second-order F and M processes are then investigated.

3 A NEW APPROACH FOR STUDYING THE STATISTICS OF MANY PROCESSES IN THE MECHANICS OF THE SEA WAVES

A new approach to investigate the statistical properties of many second-order narrow-band processes in the mechanics of the sea waves was given by Arena & Fedele (2002). They defined the family ψ of stochastic processes, with (x,y) parameters:

$$\psi(x,y,t) = f(x,y)a\cos[\chi(t)] + g(x,y)a^2\cos^2[\chi(t)] + h(x,y)a^2\sin^2[\chi(t)], \qquad (1)$$

where a is a stochastic variable with Rayleigh distribution and where

$$\chi(t) = \omega_0 t + \vartheta, \qquad (2)$$

being ω_0 the angular frequency, t the time and ϑ a stochastic variable uniformly distributed in $(0,2\pi)$. By defining the two independent stochastic Gaussian processes

$$Z_1 = a\cos\chi/\sigma, \qquad Z_2 = a\sin\chi/\sigma, \qquad (3)$$

where σ^2 is the variance of both the linear processes $a\cos\chi$ and $a\sin\chi$, the Eq. (1) may be rewritten as

$$\psi(Z_1, Z_2) = \sigma\left[f(x,y)Z_1 + \sigma g(x,y)\,Z_1^2 + \sigma h(x,y)Z_2^2\right]. \qquad (4)$$

Arena & Fedele (2002) obtained that the statistical properties of the stochastic family (1) (or (4)) depend upon the deterministic parameters α_1, α_2, which are defined as:

$$\alpha_1 = \sigma g/|f|, \qquad \alpha_2 = \sigma h/|f|. \qquad (5)$$

In order to determine the distributions of the absolute maximum and of the absolute minimum, let us rewrite Eq. (1) in the form:

$$\psi(x,y) = f(x,y)a\cos(\chi) + [g(x,y) - h(x,y)]\frac{a^2}{2}\cos(2\chi) + [g(x,y) + h(x,y)]\frac{a^2}{2}. \qquad (6)$$

Supposing that $f > 0$, from Eq. (6) we obtain the amplitudes of the absolute maximum and of the absolute minimum (in absolute value), which are given respectively by:

$$\Psi_{high} = f(x,y)a + g(x,y)a^2, \quad \Psi_{low} = f(x,y)a - g(x,y)a^2. \tag{7}$$

Arena and Fedele (2002) showed that if the condition

$$|f(x,y)| / \{ |g(x,y) - h(x,y)| \, a/2 \} > 4 \tag{8}$$

is satisfied, the probabilities of exceedance $P(\xi_{high} > \xi)$ of the absolute maximum and $P(\xi_{low} > \xi)$ of the absolute minimum of the dimensionless variables

$$\xi_{high} = \frac{\Psi_{high}}{\sigma_\psi} = u\beta + \alpha_1 \beta u^2, \quad \xi_{low} = \frac{\Psi_{low}}{\sigma_\psi} = u\beta - \alpha_1 \beta u^2, \tag{9}$$

where the random variable *u* has Rayleigh distribution, are given by:

if $\alpha_1 > 0$:

$$\begin{cases} P(\xi_{high} > \xi) = \varphi_1(\xi) \\ P(\xi_{low} > \xi) = \begin{cases} \varphi_2(\xi) & \text{if } \xi \le \beta/(4\,|\alpha_1|) \\ 0 & \text{if } \xi > \beta/(4\,|\alpha_1|) \end{cases} \end{cases}$$

if $\alpha_1 < 0$:

$$\begin{cases} P(\xi_{high} > \xi) = \begin{cases} \varphi_2(\xi) & \text{if } \xi \le \beta/(4\,|\alpha_1|) \\ 0 & \text{if } \xi > \beta/(4\,|\alpha_1|) \end{cases} \\ P(\xi_{low} > \xi) = \varphi_1(\xi) \end{cases} \tag{10}$$

where

$$\varphi_1(\xi) = \exp\left[-1/(8\alpha_1^2)\left(1 - \sqrt{1 + 4|\alpha_1| \xi / \beta}\right)^2 \right],$$

$$\varphi_2(\xi) = \exp\left[-1/(8\alpha_1^2)\left(1 - \sqrt{1 - 4|\alpha_1| \xi / \beta}\right)^2 \right] - \exp\left[-1/(8\alpha_1^2)\left(1 + \sqrt{1 - 4|\alpha_1| \xi / \beta}\right)^2 \right], \tag{11}$$

$$\beta = \left(1 + 2\alpha_1^2 + 2\alpha_2^2\right)^{-1/2}.$$

Condition (8) is satisfied for $|\alpha_1 - \alpha_2| \le .135$, with probability failure smaller than 1/1000; as consequence the probabilities of exceedance (10) are good approximations of the exact probabilities of exceedance of the non-linear process (1) (Arena & Fedele, 2002).

4 THE RANDOM WAVE FIELD IN FRONT OF A VERTICAL WALL

Let us assume Rectangular Cartesian co-ordinates having the *x*-axis horizontal and the vertical *y*-axis with origin at the mean water level. A vertical wall is located at the abscissa $x = 0$. The incident waves move along the *x*-axis.

To the second-order in a Stokes expansion the narrow-band free surface displacement in front of a vertical wall is given by

$$\eta(x,t) = 2\sigma\cos(kx)Z_1 + 2\sigma\varepsilon\cos(2kx)(f_{\eta_1}+f_{\eta_2})Z_1^2 + 2\sigma\varepsilon\cos(2kx)(-f_{\eta_1}+f_{\eta_2})Z_2^2 \quad (12)$$

where

$$f_{\eta_1}(kd) = [2+\cosh(2kd)]\cosh(kd)/[4\sinh^3(kd)]\,,\quad f_{\eta_2}(kd) = [2\tanh(2kd)]^{-1}\,,\quad (13)$$

and the narrow-band second-order wave pressure Δp (being Δp the difference between the actual pressure p and the pressure at rest) is given by

$$\frac{\Delta p(x,y,t)}{\rho g} = 2\,\sigma\, f_{ph_1}\cos(kx)\,Z_1 + 2\,\sigma\varepsilon[f_{ph_2}\cos(2kx) - f_{ph_3} + f_{ph_4} + \\ + f_{ph_5}\cos(2kx)]Z_1^2 - 2\,\sigma\varepsilon[f_{ph_2}\cos(2kx) + f_{ph_3} + f_{ph_4} - f_{ph_5}\cos(2kx)]Z_2^2\,, \quad (14)$$

where

$$f_{ph_1}(ky,kd) = \frac{\cosh[k(y+d)]}{\cosh(kd)}\,,\quad f_{ph_2}(ky,kd) = \frac{3\cosh[2k(y+d)] - \sinh^2(kd)}{4\sinh^3(kd)\cosh(kd)}\,,$$

$$f_{ph_3}(ky,kd) = \frac{\cosh[2k(y+d)]-1}{2\sinh(2kd)}\,,\ f_{ph_4}(ky,kd) = \frac{\cosh^2[k(y+d)] - \cosh(2kd)}{\sinh(2kd)}\,,\quad (15)$$

$$f_{ph_5}(kd) = \frac{1}{2\sinh(2kd)}\,.$$

We have that both the free surface displacement [Eq. (12)] and the fluctuating wave pressure [Eq. (14)] belong to the stochastic family (1). The parameters α_1 and α_2 [see Eq. (5)] of either the free surface displacement or the fluctuating wave pressure on the vertical wall are markedly greater than the corresponding parameters obtained in an undisturbed wave field, as was pointed out by the authors (Arena & Fedele, 2002). As a consequence the effects of non-linearity of both the free surface displacement and of the fluctuating wave pressure on a vertical wall are stronger than in the undisturbed field.

4.1 The narrow-band second-order wave force on the vertical wall

The wave force on the vertical wall is given by

$$F(t) = \int_{-d}^{\eta(0,t)} p(0,y,t)\,\mathrm{d}y - \int_{-d}^{0} p_h(y)\,\mathrm{d}y\,, \quad (16)$$

where p_h is the hydrostatic pressure. For $\eta(0,t)$>0 Eq. (16) may be rewritten as

$$F_+(t) = \int_{-d}^{0} \Delta p(0,y,t)\,\mathrm{d}y + \int_{0}^{\eta(0,t)} \Delta p(0,y,t)\,\mathrm{d}y \quad (17)$$

where in the first integral Δp gives the difference between actual pressure p and p_h. The last integral, by assuming that the wave pressure has linear variation between y=0 and $\eta(0,t)$, is evaluated as $\Delta p(0,0,t)\,\eta(0,t)/2 = 2\rho g\sigma^2 Z_1^2$ (let us note that pressure at rest is zero for $\eta > 0$). For $\eta(0,t)$<0 Eq. (16) may be rewritten as

$$F_-(t) = \int_{-d}^{0} \Delta p(0,y,t)\,\mathrm{d}y - \int_{-|\eta(0,t)|}^{0} p(0,y,t)\,\mathrm{d}y\,, \tag{18}$$

where p, that gives the actual pressure under the mean water level ($\eta \le 0$), is given by $\Delta p + p_h$. The last integral in Eq. (18) retaining the second order term only, gives the value $-2\rho g\sigma^2 Z_1^2$. As a consequence the wave force (16) at any instant t (and therefore for any η) may be rewritten as:

$$F(t) = \int_{-d}^{0} \Delta p(0,y,t)\,\mathrm{d}y + 2\rho g\sigma^2 Z_1^2 + o(a^2)\,, \tag{19}$$

where $o(a^2)$ includes all the higher order terms neglected. By considering the wave pressure given by Eq. (14) and solving the integral we obtain the narrow-band second-order random force on the vertical wall, that is given by

$$\frac{F(t)}{\rho g d\sigma} = f_0\, Z_1 + \varepsilon f_1 Z_1^2 + \varepsilon f_2 Z_2^2 \tag{20}$$

being ε the wave steepness (characteristic values of the wave steepness are over the range $0.05 < \varepsilon < 0.06$) and

$$\begin{gathered} f_0(kd) = 2\tanh(kd)/(kd), \qquad f_1(kd) = 2\left\{3/[4kd\sinh^2(kd)] - \tanh(kd) + \underline{1/kd}\right\}, \\ f_2(kd) = -1/2\left\{1/kd - 4/\tanh(kd) + 1/[kd\tanh^2(kd)] + 2/[kd\sinh^2(kd)]\right\}. \end{gathered} \tag{21}$$

Let us note that underlined term [that is $2\varepsilon Z_1^2/(kd)$], gives a positive contribution to the wave force, at any instant t. The integral of Eq. (13) has a positive value for the wave crest (positive peak of the wave force) and a negative value for the wave trough (negative peak of the wave force). Therefore the second-order force $F(t)$ [Eq. (20)] belongs to the non-linear stochastic family (1) with parameters:

$$\alpha_1 = \varepsilon f_1/f_0 \qquad \alpha_2 = \varepsilon f_2/f_0\,. \tag{22}$$

Fig. 1 shows the parameters α_1 and α_2 of the random force process $F(t)$, as a function of the bottom depth kd. The probabilities of exceedance of positive peaks and of negative peaks are therefore given by Eq. (10): they well approximate the exact probabilities of exceedance of the non-linear stochastic family (1) when $|\alpha_1 - \alpha_2|$<0.135. It is easy to verify that for the non-linear $F(t)$ [Eq. (20)], assuming ε=0.055, this inequality is satisfied if 0.8<kd<2.0 (see Fig. 1). We have α_1 <0 for kd>1.38: the wave force has positive peaks smaller than negative peaks. Let us note that the wave force peaks are positive (negative) when a wave crest (wave trough) of the free surface displacement on the wall occurs. For kd=1.38 the force is a quasi-symmetric processes ($\alpha_1 = 0$) and therefore the nonlinearities are weak: both the positive peak distribution and the negative peak distribution of the wave force are given by the Rayleigh law with good approximation. For kd<1.38 we have α_1 >0; in this case we have positive peaks greater than negative peaks.

This asymmetry of the wave force process is important for some applications in ocean engineering. For example by calculating the wave force on a vertical breakwater, according to the first order in a Stokes expansion we obtain that the maximum force [produced by a wave crest of the $\eta(0,t)$ process, see Eq. (12)] is equal to the absolute value of the minimum force [produced by a wave trough of the $\eta(0,t)$ process].

The narrow-band second-order wave force process (20) gives a more accurate model for the wave force on a vertical wall: while the linear wave force presents always positive peaks equal to the absolute value of negative peaks, the second-order wave force has a characteristic asymmetry between positive and negative peaks depending upon the bottom depth (see Fig. 1); in particular we found that for kd>1.38 (kd<1.38) the absolute value of the negative peaks of the wave force are greater (less) than the positive peaks. Condition of quasi-symmetry is realized for kd close to 1.38.

These analytical results are in good agreement with experimental evidence by Arena (1995) who analyzed the force of sea waves on a vertical wall during a small-scale field experiment (see Boccotti, 2000). The bottom depth at the wall was 1.45m, with kd values between 1.23 and 1.65. In particular he compared the maximum force $F_{\max}$ and the minimum force (in absolute value) $F_{\min}$ during 51 records, each of which composed by 250-300 waves. Fig. 3 shows the experimental values of the ratio $F_{\max} / F_{\min}$ during each record, which agree with our analytical predictions.

4.2 The narrow-band second-order overturning moment on the vertical wall

The overturning moment (with respect to the bottom depth), by retaining terms up to the second order, is given by

$$M(t) = F(t)d + \int_{-d}^{0} \Delta p(0,y,t)\, y \,\mathrm{d}y + o(a^2)\,, \tag{23}$$

where $o(a^2)$ includes all the higher order terms neglected. By considering the fluctuating wave pressure given by Eq. (14) the narrow-band second-order overturning moment is given by:

$$\frac{M(t)}{\rho g d^2 \sigma} = m_0\, Z_1 + \varepsilon\, m_1 Z_1^2 + \varepsilon\, m_2 Z_2^2 \tag{24}$$

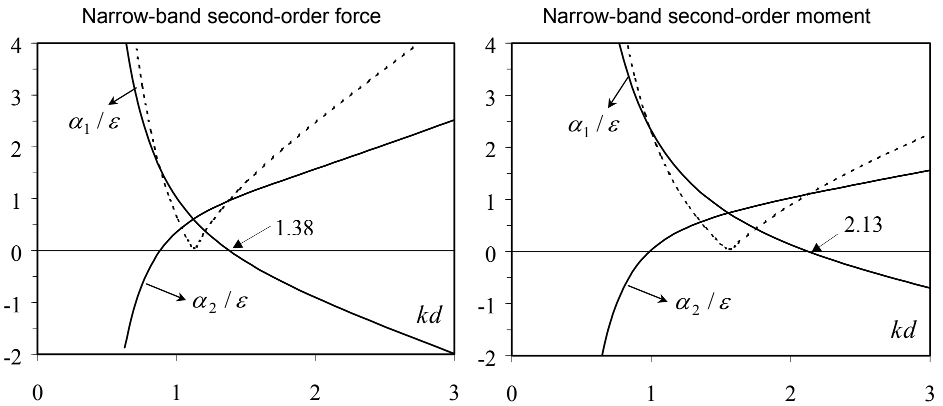

Figure 1. The non-linearity parameters α_1 and α_2 of the narrow-band second-order wave force (left panel) and overturning moment (right panel) on a vertical wall, as a function of the bottom depth kd . Dashed lines give the $\left|\alpha_1 - \alpha_2\right| / \varepsilon$ values.

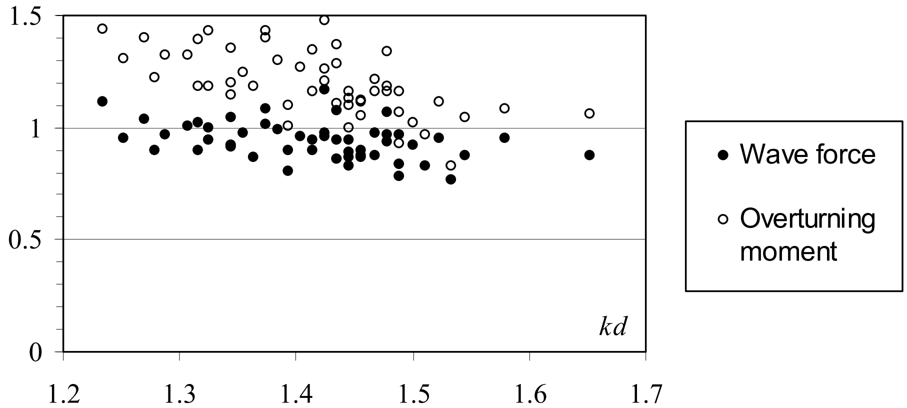

Figure 2. The experimental $F_{\max} / F_{\min}$ and $M_{\max} / M_{\min}$ values during records with 250-300 waves of a small-scale field experiment (Boccotti, 2000). Data from Arena (1995).

with

$$
\begin{aligned}
m_0(kd) &= 2(kd)^{-2}\left\{-1+\left[\cosh(kd)\right]^{-1}+kd\tanh(kd)\right\},\\
m_1(kd) &= 1/4(kd)^{-2}\left\{-3/\tanh(kd)+6kd/\sinh^2(kd)+\left[3-4(kd)^2\right]\tanh(kd)+8kd\right\},\\
m_2(kd) &= -1/4(kd)^{-2}\left\{4kd-\left[3+4(kd)^2\right]/\tanh kd+6kd/\sinh^2(kd)+\tanh(kd)\right\}.
\end{aligned}
\quad (25)
$$

Therefore the second-order $M(t)$, given by Eq. (24), belongs to the non-linear stochastic family (1) with parameters:

$$\alpha_1 = \varepsilon\, m_1 / m_0 \qquad \alpha_2 = \varepsilon\, m_2 / m_0 \quad . \tag{26}$$

Fig. 1 shows the parameters α_1 and α_2 of $M(t)$ process [Eq. (26)] as a function of the bottom depth kd. Assuming $\varepsilon = 0.055$, the inequality $|\alpha_1 - \alpha_2|<0.135$ is satisfied if $kd>0.98$.

As we can see α_1 has positive values for $kd<2.13$ and negative values for $kd>2.13$. Let us note that the threshold $kd=2.13$ for which $M(t)$ is a quasi-symmetric process is greater than the corresponding threshold of the wave force $F(t)$ (that is $kd=1.38$). This is congruent from a physical point of view: for $kd=1.38$ the wave force is quasi-symmetric and has positive peaks F_{max} very close to negative peaks F_{min} (in absolute value). Because F_{max} is produced by a crest of process $\eta(0,t)$ [see Eq. (12)] and F_{min} is produced by a trough of $\eta(0,t)$, for $kd=1.38$ the moment of the wave force F_{max} has to be greater than the moment of the wave force F_{min}. That is for $kd=1.38$ (see right panel in Fig. 1) we have $\alpha_1>0$ for the overturning moment (24), which implies positive $M(t)$ peaks greater than negative $M(t)$ peaks. This is in agreement with the experimental values of M_{max}/ M_{min} showed in Fig. 3 (data by Arena, 1995).

5 References

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