Tail probabilities of Successive Wave Crest Heights in Gaussian seas

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Abstract

The theory of quasi-determinism introduced by Boccotti gives necessary and sufficient conditions for the occurrence in a gaussian sea state of a wave of very large given height. As a corollary, Boccotti derives that the probability of exceedance of the wave height follows asymptotically a Weibull distribution which depends upon the narrow bandedness parameter ψ^*defined as the absolute value of the quotient between the first absolute minimum and the absolute maximum of the autocovariance function $\psi(T)$. In this paper, the necessary and sufficient conditions for the occurrence, in gaussian sea states, of two successive wave crests of large heights are given. As a corollary it is proven that the tail probability of the joint distribution of two successive wave crests is given by a bivariate Weibull law, which depends upon the parameter $\psi_2^* = \psi(T_2^*)/\psi(0)$ with T_2^* the abscissa of the second absolute maximum of $\psi(T)$. The analytical results are in agreement with Monte Carlo simulations.

Key words: Successive wave crests; gaussian sea; quasi-determinism; bivariate Weibull; conditional probability.

1 Introduction

The theory of quasi-determinism for the mechanics of linear wave groups was derived by Boccotti in the eighties, with two formulations. The first one ([1],[2]) enables us to predict what happens when a very high crest occurs in a fixed time and location (see also [3]); the second one ([4],[5]) gives the mechanics of the wave group when a very large crest-to-trough height occurs. The theory, which is exact to the first order in a Stokes expansion (Gaussian sea), is valid for any boundary condition (for example either for waves in an undisturbed field or in reflection). The theory was then verified in the nineties with some small-scale field experiments ([6],[7]).

Following Boccotti ([5]), the necessary and sufficient condition for the occurrence of two successive wave crests of very large given height is provided and as a corollary, it is proven that the tail probability of the joint distribution of two successive wave crests is given by a bivariate Weibull distribution. The analytical prediction are in good agreement with Monte Carlo simulations.

2 The theory of quasi-determinism

2.1 Gaussian sea states

According to the theory of sea states, to the first order in a Stokes' expansion, a time series of surface displacements $\eta(t)$ recorded at a fixed point at sea is a realization of a stationary ergodic stochastic gaussian process defined as

$$\eta(t) = \sum_{i=1}^{N} a_i \cos(\omega_i t + \varepsilon_i) \tag{1}$$

where it is assumed that frequencies ω_i are different from each other, the number N is infinitely large, phase angles ε_i are uniformly distributed on $[0, 2\pi]$ and are stochastically independent of each other, and all the amplitudes a_i satisfy the frequency spectrum $S(\omega)$ defined as

$$S(\omega)d\omega = \sum_{\omega_i \in [\omega, \omega + d\omega]} \frac{a_i^2}{2}. \tag{2}$$

The jth order moment of the spectrum is $m_j = \int_0^\infty S^j(\omega)d\omega$. In particular $m_0 = \sigma^2$, where σ is the standard deviation of $\eta(t)$. The autocovariance function $\psi(T)$ can be evaluated as $\psi(T) = \int_0^\infty S(\omega)\cos(\omega T)\ d\omega$.

2.2 The occurrence of a wave of large height

Let us consider the surface displacement $\eta(t)$ at any fixed point (x_0, y_0) in a random wave field. Setting t_0 as an arbitrary time instant, H the wave height and T^* the abscissa of the absolute minimum of the autocovariance function, Boccotti showed that the condition

$$\eta(t_0) = \frac{H}{2} \quad \eta(t_0 + T^*) = -\frac{H}{2} \tag{3}$$

becomes necessary and sufficient for the occurrence of a wave of height H as $\alpha = H/\sigma \to \infty$. The condition (3) is sufficient because as $\alpha \to \infty$ the conditional probability

$$\Pr\left[\eta(t_0 + T) = u \Big/ \eta(t_0) = \frac{H}{2}, \eta(t_0 + T^*) = -\frac{H}{2}\right] \tag{4}$$

tends to a delta function $\delta[u - \bar{\eta}(t_0 + T)]$ centered at

$$\bar{\eta}(t_0 + T) = \frac{H}{2}\frac{\psi(T) - \psi(T - T^*)}{\psi(0) - \psi(T^*)} \tag{5}$$

Implying that as $\alpha \to \infty$, given condition (3), with probability approaching one, the surface displacement $\eta(t_0 + T)$ tends to the deterministic form $\bar{\eta}(t_0 + T)$. This is a wave profile with wave height H, having a crest of amplitude $H/2$ at $T = 0$ and a trough of amplitude $H/2$ at $T = T^*$.

In order to show that Eq. (3) is also a necessary condition, Boccotti derived the analytical expression for the expected number per unit time $EX(\alpha, \tau, \xi)$ of local maxima of the surface displacement $\eta(t)$ with amplitude $\xi\alpha$ which are followed by a local minimum with amplitude $(\xi - 1)\alpha$ after a time lag τ. He showed that as $\alpha \to \infty$ in the domain (τ, ξ) there exists an $O(\alpha^{-1})$ infinitesimal neighborhood of $(T^*, 1/2)$ such that

$$EX_{s.w.}(\alpha, \tau, \xi) = \begin{cases} EX(\alpha, \tau, \xi) & \tau = T^* + \delta\tau,\ \xi = \frac{1}{2} + \delta\xi \qquad (\delta\tau, \delta\xi) \sim O(\alpha^{-1}) \\ & 0 \ \ elsewhere \end{cases} \tag{6}$$

where $EX_{s.w.}(\alpha, \tau, \xi)$ is the expected number per unit time of local maxima of the surface displacement $\eta(t)$ with amplitude $\xi\alpha$ which are followed by a local minimum with amplitude

$(\xi - 1)\alpha$ after a time lag between τ,*where the local maximum and the local minimum must be, respectively the crest and the trough of the same wave* (the subscript $s.w.$ stands for *same wave*). Hence condition (3) is also necessary in the limit $\alpha \to \infty$.

As a corollary Boccotti derived the tail probability of the wave height distribution $p(\alpha)$, which has the following expression

$$p(\alpha) = 2\pi \int_0^\infty \int_0^1 EX_{s.w.}(\alpha, \tau, \xi)\, d\tau d\xi \to \frac{\alpha}{2(1+\psi^*)} \exp\left[-\frac{\alpha^2}{4(1+\psi^*)}\right] \quad as\ \alpha \to \infty$$

where the narrow-bandedness parameter ψ^* is defined as the absolute value of the quotient between the first absolute minimum and the absolute maximum of the autocovariance function.

3 The occurrence of two successive wave crests of very large given heights

3.1 Sufficient condition

In the following the theory of quasi-determinism of Boccotti is extended to study the occurrence of two very large successive wave crests. Let us consider the surface displacement $\eta(t)$ at any fixed point (x_0, y_0) in a random wave field defined as in Eq. (1). Let us analyze the probability density function of the surface displacement at any fixed time, given the condition

$$\eta(t_0) = h_1 \text{ and } \eta(t_0 + T_2^*) = h_2 \tag{7}$$

where t_0 is an arbitrary time instant, h_1 and h_2 are crest amplitudes and T_2^* is the abscissa of the second absolute maximum of the autocovariance function $\psi(T)$. The p.d.f. of $\eta(t)$ at time $t_0 + T$, given condition (7) is gaussian, i.e.

$$\Pr[\eta(t_0 + T) = u / \eta(t_0) = h_1, \eta(t_0 + T_2^*) = h_2] = \frac{1}{\sqrt{2\pi\sigma_c^2}} \exp\left\{-\frac{[u - \eta_c(t_0 + T)]^2}{2\sigma_c^2}\right\}$$

Where the conditional mean $\eta_c(t_0 + T)$ is given by

$$\eta_c(t_0 + T) = \frac{h_1\psi(0) - h_2\psi(T_2^*)}{\psi^2(0) - \psi^2(T_2^*)}\psi(T) + \frac{h_2\psi(0) - h_1\psi(T_2^*)}{\psi^2(0) - \psi^2(T_2^*)}\psi(T - T_2^*) \tag{8}$$

And the conditional variance σ_c^2 has the following expression

$$\sigma_c^2 = \sigma^2 \left[1 - \frac{\psi^2(T_2^*) + \psi^2(T - T_2^*) - 2\psi(T)\psi(T - T_2^*)\frac{\psi(T_2^*)}{\psi(0)}}{\psi^2(0) - \psi^2(T_2^*)}\right].$$

Since $\frac{\psi(T_2^*)}{\psi(0)}$ is smaller than unity, the conditional variance is always bounded by the unconditional variance σ^2. Therefore as $\frac{h_1}{\sigma} \to \infty$ and $\frac{h_2}{\sigma} \to \infty$ the ratio $\frac{\sigma_c}{\eta_c(t_0+T)}$ approaches zero (since $\eta_c(t_0 + T) \to \infty$ and σ_c is bounded by the unconditional standard deviation σ) implying that all the realizations of the random wave field which satisfy condition (7), with probability approaching one, tend to the deterministic profile $\eta_c(t_0 + T)$. The conditional mean $\eta_c(t_0 + T)$, [see Eq. (8)] represents a wave structure of two successive wave crests lagged in time by T_2^*, if

$\eta_c(t_0+T)$ attains two local maxima at $T=0, T_2^*$, i.e. if the second order derivatives at those abscissas are always less than zero

$$\ddot{\eta}_c(0) < 0 \quad \text{and} \quad \ddot{\eta}_c(T_2^*) < 0 \tag{9}$$

Some algebra yields $\ddot{\eta}_c(0) = a(-\beta_0 + s\ \beta_1)$, $\ddot{\eta}_c(T_2^*) = a(-\beta_1 + s\ \beta_0)$ with $\beta_0 = \frac{h_1}{\sigma}$ and $\beta_1 = \frac{h_2}{\sigma}$ and

$$a = \frac{1+\psi(T_2^*)\ddot{\psi}(T_2^*)}{1-\psi^2(T_2^*)} \qquad s = \frac{\psi(T_2^*)+\ddot{\psi}(T_2^*)}{1+\psi(T_2^*)\ddot{\psi}(T_2^*)}$$

where dot denotes derivative. Since a is always greater or equal to zero, condition (9) is fulfilled if

$$\begin{cases} \beta_0, \beta_1 \in \mathbf{R}_+^2 & \text{if } s \le 0 \\ \beta_0, \beta_1 \in \Omega(s) & \text{if } s > 0 \end{cases} \tag{10}$$

where

$$\Omega(s) = \left\{ (\beta_0, \beta_1) \in \mathbf{R}_+^2 : \beta_0 \ge 0, \beta_1 \ge 0 \ \ s < \frac{\beta_1}{\beta_0} < \frac{1}{s} \right\}$$

is an open sectorial region of $\mathbf{R}_+^2$ with aperture angle $\theta = \pi/2 - 2\tan^{-1}(s)$. Furthermore because it is assumed that the autocovariance function $\psi(T)$ attains only one minimum in the open interval $(0, T_2^*)$ at $T = T^*$, the two local maxima of the wave profile $\eta_c(t_0+T)$ are also two consecutive wave crests. Hence as $\beta_0 \to \infty$ and $\beta_1 \to \infty$, condition (7) is sufficient for the occurrence of two successive wave crests of given very large height under the constraint (10).

Typical JONSWAP spectra satisfy the condition $s > 0$ with $s \in [0.14, 0.16]$. As the spectra gets narrow the sector $\Omega(s)$ tends to cover all $\mathbf{R}_+^2$, i.e. $\theta \to \pi/2$, because s approaches zero in the narrow-band limit.

3.2 The condition (7) is necessary for the occurrence of two large successive wave crests

3.2.1 The definition of the general form of $EX_c(\beta_0, \beta_1, \tau)$

In the following, the notations ψ_T, η_T are adopted to indicate respectively the autocovariance $\psi(T)$ and the surface displacement $\eta(T)$. Without loosing generality, the time scale $\frac{1}{\sqrt{m_2}}$ and the length scale $\sigma = \sqrt{m_0}$ are used to non-dimensionalize Eq. (1) such that the zero*th* and second order moment of the spectrum are unitary, i.e. $m_0 = 1$, $m_2 = 1$. It follows that $\psi_0 = 1, \ddot{\psi}_0 = -1$. Let us consider the expected number per unit time

$$EX_c(\beta_0, \beta_1, \tau)\, d\beta_0 d\beta_1 \tag{11}$$

of local maxima of the surface displacement $\eta(t)$ (at a fixed location in space) whose elevation is between β_0 and $\beta_0 + d\beta_0$, and which are followed by a local maximum with an elevation between β_1 and $\beta_1 + d\beta_1$ after a time lag between τ and $\tau + d\tau$. Following the general approach introduced by Rice (see [5], pp. 159-162), $EX_c(\beta_0, \beta_1, \tau)$ can be expressed as

$$EX_c(\beta_0, \beta_1, \tau) = \int_0^\infty \int_0^\infty |z_1 z_2|\, p\left[\eta_0 = \beta_0, \dot{\eta}_0 = 0, \ddot{\eta}_0 = z_1, \eta_\tau = \beta_1, \dot{\eta}_\tau = 0, \ddot{\eta}_\tau = z_2\right] dz_1 dz_2 \tag{12}$$

where $p\left[\eta_0, \dot{\eta}_0, \ddot{\eta}_0, \eta_\tau, \dot{\eta}_\tau, \ddot{\eta}_\tau\right]$ is a gaussian joint probability density function. Eq. (12) is rewritten in the form

$$\begin{aligned} EX_c\left(\beta_0, \beta_1, \tau\right) &= p\left[\eta_0 = \beta_0, \dot{\eta}_0 = 0, \dot{\eta}_\tau = 0, \eta_\tau = \beta_1\right] \cdot \\ &\cdot \int_0^\infty \int_0^\infty \left|z_1 z_2\right| p\left[\ddot{\eta}_0 = z_1, \ddot{\eta}_\tau = z_2 / \, \eta_0 = \beta_0, \dot{\eta}_0 = 0, \dot{\eta}_\tau = 0, \eta_\tau = \beta_1\right] dz_1 dz_2 \end{aligned} \tag{13}$$

In the limit of $\beta_0 \to \infty$ and $\beta_1 \to \infty$

$$p\left[\ddot{\eta}_0 = z_1, \ddot{\eta}_\tau = z_2 / \eta_0 = \beta_0, \dot{\eta}_0 = 0, \dot{\eta}_\tau = 0, \eta_\tau = \beta_1\right] \to \delta\left[z_1 - \ddot{\eta}_c\left(0\right), z_1 - \ddot{\eta}_c\left(T_2^*\right)\right]$$

yielding the simplification

$$EX_c\left(\beta_0, \beta_1, \tau\right) = p\left[\eta_0 = \beta_0, \dot{\eta}_0 = 0, \dot{\eta}_\tau = 0, \eta_\tau = \beta_1\right] \ddot{\eta}_c\left(0\right) \ddot{\eta}_c\left(T_2^*\right) \tag{14}$$

The joint probability $p\left[\eta_0 = \beta_0, \dot{\eta}_0 = 0, \dot{\eta}_\tau = 0, \eta_\tau = \beta_1\right]$ in Eq. (14) can be Taylor expanded with respect to the variable τ around $\tau = T_2^*$ yielding

$$EX_c\left(\beta_0, \beta_1, \tau\right) = \frac{1}{\left(2\pi\right)^2 \sqrt{1 - \psi_{T_2^*}^2}} \exp\left[-\frac{\beta_0^2 + \beta_1^2 - 2\psi_{T_2^*}\beta_0\beta_1}{2\left(1 - \psi_{T_2^*}^2\right)} + \frac{1}{2}K^* \delta\tau^2\right] + o(\delta\tau^2) \tag{15}$$

where $K^* = -\frac{\ddot{\psi}_{T_2^*}}{1 - \ddot{\psi}_{T_2^*}^2} \ddot{\eta}_c\left(0\right) \ddot{\eta}_c\left(T_2^*\right)$ can be proven to be greater than zero. Hence as $\beta_0 \to \infty$ and $\beta_1 \to \infty$ there exists an infinitesimal neighborhood $\delta\Gamma$ of order $O(K_*^{-1/2})$ such that

$$EX_c\left(\beta_0, \beta_1, \tau\right) = \begin{cases} EX_c\left(\beta_0, \beta_1, T_2^*\right) & \tau = T_2^* + \delta\tau \quad \delta\tau \in \delta\Gamma \\ 0 & elsewhere \end{cases} \qquad as \;\; \beta_0 \to \infty \text{ and } \beta_1 \to \infty. \tag{16}$$

This means that a local maxima of a very large amplitude β_0 followed by a local maxima of a very large amplitude β_1 after a time lag $T_2^* + \delta\tau$ with $\delta\tau \in \delta\Gamma$ have the same expectation as two local maxima with amplitudes respectively equal to β_0 and β_1, lagged in time by T_2^*. But two local maxima of large amplitude lagged in time by T_2^* are also two successive crests because the condition (7) is sufficient. Hence the latter condition (7) is also necessary in the limit $\beta_0 \to \infty$ and $\beta_1 \to \infty$.

3.2.2 The tail probabilities of two successive wave crest eights

Let us define

$$EX_{s.c.}\left(\beta_0, \beta_1, \tau\right) d\beta_0 d\beta_1 \tag{17}$$

The expected number per unit time of local maxima of the surface displacement $\eta\left(t\right)$ (at a fixed location in space) whose elevation falls between β_0 and $\beta_0 + d\beta_0$ and are followed by a local maxima of elevation between β_1 and $\beta_1 + d\beta_1$ after a time lag between τ and $\tau + d\tau$, *where the two local maxima must be two successive wave crests* (the subscript $s.c.$ stands for *successive crests*). As β_0 and $\beta_1 \to \infty$, from Eq. (16) two successive wave crests lagged in time by $T_2^* + \delta\tau$ with $\delta\tau \in \delta\Gamma$ are, with probability approaching one, two local maxima lagged in time by $T_2^* + \delta\tau$ yielding

$$EX_{s.c.}\left(\beta_0, \beta_1, \tau\right) = \begin{cases} EX_c\left(\beta_0, \beta_1, \tau\right) & \tau = T_2^* + \delta\tau \quad \delta\tau \in \delta\Gamma \\ 0 & elsewhere \end{cases}. \tag{18}$$

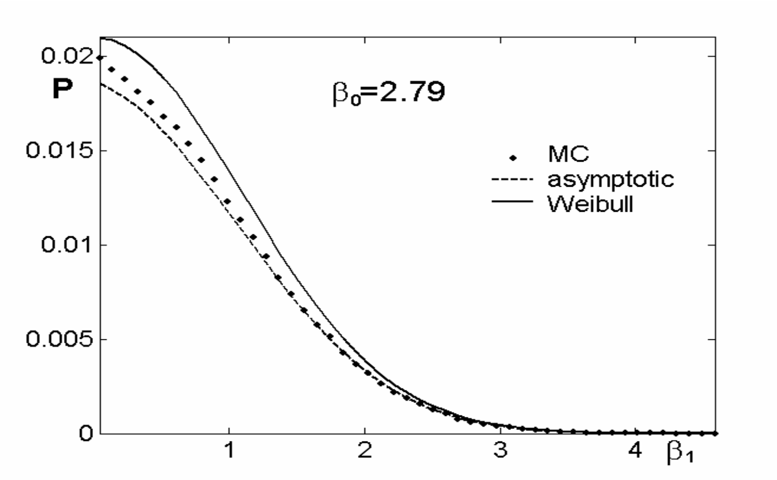

Figure 1: The probabilities of exceedance for $\beta_0 = 2.79$.

The exact expression for the joint probability density function of two successive wave crests is $p(\beta_0, \beta_1) = \frac{1}{EX_+} \int_0^\infty EX_{s.c.}(\beta_0, \beta_1, \tau)\, d\tau$ where $EX_+ = \frac{1}{2\pi}\sqrt{\frac{m_2}{m_0}} = \frac{1}{2\pi}$ is the expected number per unit time of zero up-crossing of the surface displacement. If β_0 and $\beta_1 \to \infty$, since Eq. (18) holds, we have the following

$$p(\beta_0, \beta_1) \simeq \frac{1}{2\pi} \frac{\ddot{\eta}_c(0)\, \ddot{\eta}_c(T_2^*)}{1 - \psi_{T_2^*}^2} \exp\left[-\frac{\beta_0^2 + \beta_1^2 - 2\psi_{T_2^*}\beta_0\beta_1}{2\left(1 - \psi_{T_2^*}^2\right)} \right] \int_{\delta\tau \in \delta\Gamma} \exp\left(-\frac{1}{2} K^* \delta\tau^2 \right) d(\delta\tau); \quad (19)$$

The integral that appears in Eq. (19) can bounded by $\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}K^*\delta\tau^2\right) d(\delta\tau) = \frac{\sqrt{2\pi}}{\sqrt{K^*}}$ obtaining the p.d.f.

$$p_a(\beta_0, \beta_1) = \frac{1 + \psi_2^* \ddot{\psi}_2^*}{\sqrt{-2\pi\ddot{\psi}_2^*(1 - \psi_2^{*2})^3}} \exp\left[-\frac{\beta_0^2 + \beta_1^2 - 2\psi_2^*\beta_0\beta_1}{2(1 - \psi_2^{*2})} \right] \sqrt{(-\beta_0 + s\,\beta_1)(-\beta_1 + s\,\beta_0)} \quad (20)$$

where $\psi_2^* \equiv \psi_{T_2^*}$, $\ddot{\psi}_2^* \equiv \ddot{\psi}_{T_2^*}$. From Eq. (20) the following upper bound for $p_a(\beta_0, \beta_1)$ is readily derived

$$p_W(\beta_0, \beta_1) = \frac{\beta_0\beta_1}{(1 - k^2)} \exp\left[-\frac{\beta_0^2 + \beta_1^2}{2(1 - k^2)} \right] I_0\left(\frac{k\,\beta_0\beta_1}{1 - k^2} \right) \quad (21)$$

where here $k = \psi_2^*$ and $I_0(x)$ is the modified Bessel function; Eq. (21) is a bivariate Weibull distribution, used by many authors to model the distribution of successive wave heights in narrow-band gaussian seas ([8],[9]).

4 Validation

In this section the probability laws $p_a(\beta_0, \beta_1)$ and $p_W(\beta_0, \beta_1)$, i.e. Eqs. (20) and (21), are validated by performing Monte Carlo simulations with rectangular spectrum. By assuming $\omega_{\max} = 1.5$, $\omega_{\min} = 0.5$, by means of Eq. (1), realizations of a Gaussian sea state have been generated, with roughly 90000 waves. In figures 1, the theoretical probabilities of exceedance $\Pr[\beta_0 > x_0, \beta_1 > x_1]$ of the asymptotic p.d.f. (20) and the Weibull p.d.f. (21) are compared to the probabilities of exceedance derived from the Monte Carlo simulations. As one can see from the plots, the asymptotic $p_a(\beta_0, \beta_1)$ and the Weibull $p_W(\beta_0, \beta_1)$ are respectively a lower bound and an upper bound of the exact p.d.f. $p(\beta_0, \beta_1)$. The distribution $p_a(\beta_0, \beta_1)$ converges to the exact distribution $p(\beta_0, \beta_1)$ for $\beta_0 > 2$ and $\beta_1 > 2$, whereas the convergence of $p_W(\beta_0, \beta_1)$ is attained for $\beta_0 > 2.5$ and $\beta_1 > 2.5$.

5 Conclusions

Based on the theory of quasi-determinism of Boccotti, the necessary and sufficient conditions for the occurrence of two very large successive wave crests are given. As a corollary it is proven that the tail probability of the joint distribution of two successive wave crests is given by a bivariate Weibull law, where the Weibull parameter is equal to $\psi_2^* = \psi(T_2^*)/\psi(0)$ with T_2^* the abscissa of the second absolute maximum of the autocovariance function $\psi(T)$. The theoretical results agree well with the Monte Carlo simulations.

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