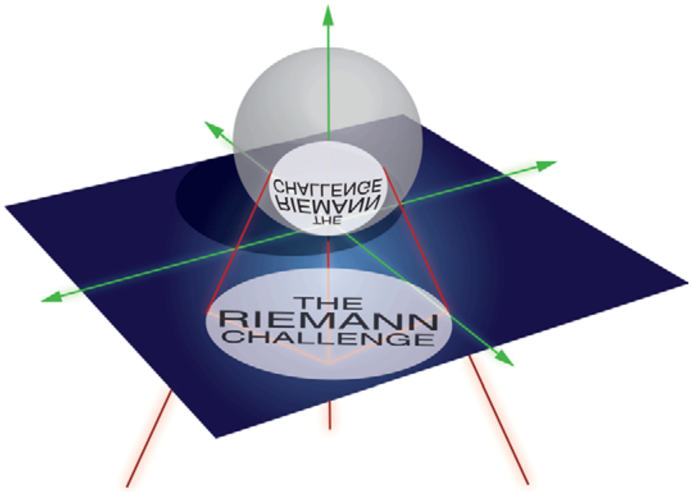
“Franz & professor Integralus”



“Franz is a German exchange student at GATECH. His supervisor is Prof. Integralus from the math department. Frank is taking Calc 3 and needs to solve the following assignment for that course:

“Consider α as a real number greater or equal to zero. Derive the function J(α) where

$$J(\alpha)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+t^2)}\,\frac{\sin[\alpha(x-t)]}{x-t}\,dx\,dt\,.$$

Show all the necessary steps and calculations by hand.”

At first sight, the assignment appears difficult, so Franz decides to ask for some help to prof. Integralus. Franz is at his office, with the assignment in hand. The following conversation takes place:
Franz: “ *I have difficulties in solving this problem, can you please help me?*”
Prof. Integralus: “*Yes I can. The problem is simple, but you should think simpler*”.
Franz: “*I am confused ??!?, what do you mean ?*”
Prof. Integralus: “*Well, you are seeking a function J(α) which just depends upon α*”
Franz: “*Ok, but this does not make things simpler at all !*”
Prof. Integralus : “*Not really. Usually functions satisfy differential equations with proper initial conditions, but now I have to go. Have a good day, Franz !*”
Prof. Integralus ends his conversation with the student. After some thoughts Franz is able to solve his assignment.
Can you also solve Franz’s integral?

Be the first to solve the Riemann challenge problem correctly, and email your solution to Dr. Fedele (ffedele3@gtsav.gatech.edu) by Monday, March 16. *Good Luck* !

Sincerely, the Riemannians

Riemann Problem - 3

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given integral is of the form

$$J(\alpha) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-1/2(x^2+t^2)} \frac{\sin[\alpha(x-t)]}{(x-t)}\, dx\, dt$$

given $\alpha \geq 0$ —— (I)

$J(0) = 0$ —— (II)

Since $J(\alpha)$ is a function of only α, $J(\alpha)$ can be differentiated w.r.t (α). Hence

$$\frac{dJ}{d\alpha} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-1/2(x^2+t^2)} \frac{\cos[\alpha(x-t)]\,\cancel{(x-t)}}{\cancel{(x-t)}}\, dx\, dt$$

$$\Rightarrow \frac{dJ}{d\alpha} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-1/2(x^2+t^2)} \left[\cos(\alpha x)\cos(\alpha t) + \sin(\alpha x)\sin(\alpha t)\right] dx\, dt$$

$$= \int_{-\infty}^{\infty} e^{-x^2/2} \cos(\alpha x)\, dx \int_{-\infty}^{\infty} e^{-t^2/2} \cos(\alpha t)\, dt + \int_{-\infty}^{\infty} e^{-x^2/2} \sin(\alpha x)\, dx \int_{-\infty}^{\infty} e^{-t^2/2} \sin(\alpha t)\, dt$$

[∵ Since x and t are independent]

Standard result

$$\int_{-\infty}^{\infty} e^{-ax^2} \cos(kx)\, dx = \sqrt{\frac{\pi}{a}}\; e^{-k^2/4a}$$

$$= \left[\sqrt{2\pi}\, e^{-\alpha^2/2}\right]\left[\sqrt{2\pi}\, e^{-\alpha^2/2}\right] + \int_{-\infty}^{\infty} e^{-x^2/2} \sin(\alpha x)\, dx \int_{-\infty}^{\infty} e^{-t^2/2} \sin(\alpha t)\, dt$$

$\because$ Since $e^{-x^2/2} \sin(\alpha x)$ and $e^{-t^2/2} \sin(\alpha t)$ are odd functions,

$\int_{-\infty}^{\infty}$ of both functions will be zero.

$$\Rightarrow \frac{dJ}{d\alpha} = 2\pi e^{-\alpha^2}$$

$$\Rightarrow J = 2\pi \int_0^{\alpha} e^{-\alpha^2}\, d\alpha \qquad [\text{since } \alpha \geq 0]$$

$$= 2\pi \left[\frac{\sqrt{\pi}}{2}\, \text{erf}(\alpha)\right]$$

$$= 2\pi \frac{\sqrt{\pi}}{2}\, \text{erf}(\alpha) \qquad [\because \text{erf}(\infty) = 1]$$

$$\boxed{J(\alpha) = \pi^{3/2}\, \text{erf}(\alpha)}$$

(Ans)

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Franz & Professor Integralus

$$\text{The function} \quad J(\alpha) = \pi \int_{-\alpha}^{+\alpha} e^{-\alpha'^2}\, d\alpha' = \pi\sqrt{\pi}\,\text{erf}(\alpha)$$

Derivation:

Intuitively we need to rotate the coordiante axis clockwise by $\pi/4$ radians, so that the new y-axis lines up along $x = t$ in the original coordinate axis, and we can use the symmetry of the problem to solve for $J(\alpha)$. Mathematically, we make the substitution:

$$p = \frac{x - t}{2}, \text{ and } \; q = \frac{x + t}{2}$$

Using the determinant of the Jacobian, $dx\, dt = 2\, dp\, dq$, and $p^2 + q^2 = (x^2 + t^2)/2$. The integral now becomes:

$$J(\alpha) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(p^2+q^2)} \frac{\sin(2\alpha p)}{2p}\, 2dq dp$$

Using $\sigma = 1/\sqrt{2}$ in the Gaussian integral we have $\int_{-\infty}^{+\infty} e^{-q^2} dq = \sqrt{\pi}$. Integrating over q, and setting $y = 2p$ we get:

$$J(\alpha) = \sqrt{\pi} \int_{-\infty}^{+\infty} e^{-y^2/4} \frac{\sin(\alpha y)}{y}\, dy \tag{1}$$

Let us define $I(\alpha) = J(\alpha)/\sqrt{\pi}$ for convenience. Differentiating $I(\alpha)$ twice with respect to α while keeping everything else constant we get:

$$I'(\alpha) = \int_{-\infty}^{+\infty} e^{-y^2/4} \frac{d}{d\alpha} \frac{\sin(\alpha y)}{y}\, dy = \int_{-\infty}^{+\infty} e^{-y^2/4} \cos(\alpha y)\, dy$$

$$\text{and} \qquad I''(\alpha) = -\int_{-\infty}^{+\infty} e^{-y^2/4}\, y \sin(\alpha y)\, dy$$

$I'(\alpha)$ can be integrated by parts with respect to y to obtain:

$$I'(\alpha) = e^{-y^2/4} \frac{\sin(\alpha y)}{\alpha}\bigg|_{-\infty}^{+\infty} + 2 \int_{-\infty}^{+\infty} e^{-y^2/4} \frac{y \sin(\alpha y)}{\alpha}\, dy$$

The first term vanishes, and the second term is equal to $-\frac{2}{\alpha} I''(\alpha)$ giving us:

$$I''(\alpha) = -\frac{\alpha}{2} I'(\alpha)$$

By inspection we can see that $I'(\alpha) = Ae^{-\alpha^2}$ satisfies the above differential equation, where A is some constant. Integrating I' with respect to α, we get:

$$I(\alpha) = A \int_{-\alpha}^{+\alpha} e^{-\alpha'^2} \, d\alpha' \tag{2}$$

Applying the limit $\alpha \to \infty$ in equation (1) makes the second term inside the integral behave as a delta function:

$$\lim_{\alpha \to +\infty} J(\alpha) = \sqrt{\pi} \int_{-\infty}^{+\infty} e^{-y^2/4} \, \pi \, \delta(y) \, dy = \pi \sqrt{\pi}$$

Applying this limit also to equation (2) gives us $A = \sqrt{\pi}$ with the final solution being:

$$J(\alpha) = \pi \int_{-\alpha}^{+\alpha} e^{-\alpha'^2} \, d\alpha'$$