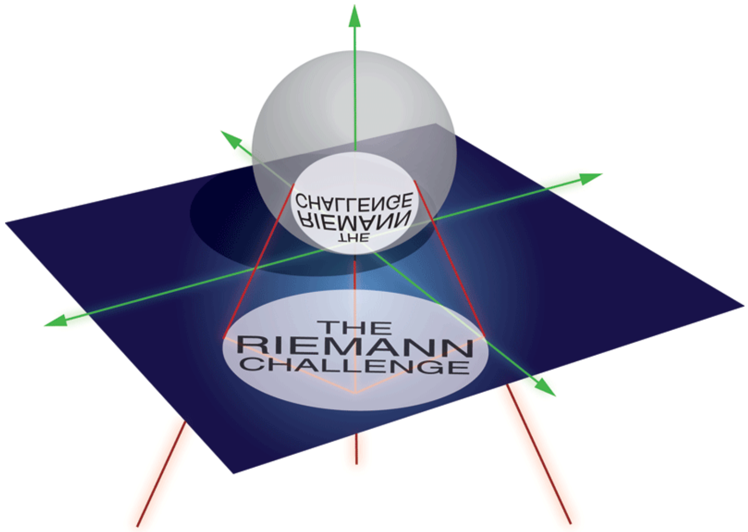
“The burning candle”



A candle in the shape of a truncated right circular cone 45 cm high burns to extinction in nine hours. The bottom 3 cm takes 20 minutes longer to burn than the top 3 cm. How many minutes will it take the top 3 cm to burn? Assume that the volume of the candle decreases at a constant rate as it burns.

Be the first to solve the Riemann challenge problem correctly, and email your solution to Dr. Fedele (ffedele3@gtsav.gatech.edu) by Friday, March 6 2009. *Good Luck* !

Rules and more:
http://www.gtsav.gatech.edu/people/ffedele/Research/riemann_challenge/index.html

Sincerely, the Riemannians

SOLUTION

Let r_t and r_b be the top and bottom radii of the candle in cm. Then, the radius at any distance from the top is given by $r = r_t + ay$, where $a = (r_b - r_t)/45$, and y is the distance from the top in cm. The equation for the volume of the frustum of a right circular cone is $V = \pi\ (h/3)(r^2 + rR + R^2)$, where r and R are the radii of the top and bottom and h is the altitude. Thus, we have

$$V_c = 15\pi[r_t^2 + r_t(r_t + 45a) + (r_t + 45a)^2] = 15\ \pi\ (3r_t^2 + 135ar_t + 2{,}025a^2),$$

$$Vt = \pi\ [r_t^2 + r_t(r_t + 3a) + (r_t + 3a)^2] = \pi\ (3r_t^2 + 9ar_t + 9a^2),$$

and

$$V_b = \pi\ [(r_t + 42a)^2 + (r_t + 42a)(r_t + 45a) + (r_t + 45a)^2] = \pi\ (3r_t^2 + 261ar_t + 5{,}679a^2).$$

Since burning time is proportional to volume, $(t_b - t_t)/t_c = (V_b - V_t)/V_c$. Using $t_b - t_t = 20$ min., $t_c = 540$ min., and the values for V_c, V_b, and V_t and making the substitution $z = rt/a$, we get $5z^2 - 531z - 13{,}635 = 0$, which can be solved by the quadratic formula to give $z = 127.576$, which allows the calculation of V_t/a^2 and V_c/a^2. Then, from the relationship $t_t = t_c(V_t/V_c)$ we can calculate $t_t = 26.433$ min, the time for the top 3 cm to burn.

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The Burning Candle

The top 3cm of the candle will burn in **26.433** minutes.

Here's how:

The volume of a right circular cone of height h is $\frac{1}{3}\pi r^2 h$ where r is the radius of the base. Since it is a right circular cone the ratio of the radius to height is a constant for any truncated parts of the cone. Let us define a "burn rate" constant a that relates the height of a full right circular cone to the number of minutes it burns.

$$h^3 = ma$$

where m is the number of minutes the candle of height h burns, and a is a lumped constant that accounts for the volume, burn rate and the ratio of the base radius to height.

Now, let h be height of the untruncated cone. $h - 45$ is the height of the part that is cut off from the given candle. So the remaining volume burns in 540 minutes. This gives us

$$h^3 - (h-45)^3 = 540a \tag{1}$$

Next, the bottom 3cm of the candle takes 20 minutes longer than the top 3cm. Volume of the top 3cm is $(h-42)^3 - (h-45)^3$, and volume of the bottom 3cm is $h^3 - (h-3)^3$. From this we get

$$h^3 - (h-3)^3 - (h-42)^3 + (h-45)^3 = 20a \tag{2}$$

Dividing eqns (2) by (1), expanding and simplifying we get

$$\frac{84h - 1890}{h^2 - 45h + 675} = \frac{5}{9}$$
$$5h^2 - 2313h + 51705 = 0$$
$$h = 172.5756\,\text{cm}.$$

and from eqn. (1)

$$a = \frac{h^3 - (h-45)^3}{540} = 5672.85$$

Time it takes for the top 3cm of the candle to burn is

$$\frac{(h-42)^3 - (h-45)^3}{a} = 26.433\,\text{minutes}$$