A complete set of eigenfunctions for the stability of pulsatile pipe flow

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Abstract

The study of pulsatile tube flow appears to have been first considered in the context of arterial hemodynamics in the mid-1950s. Womersley and co-workers ([1],[2]) obtained an exact solution of the Navier-Stokes equations for the fully-developed velocity profile of a oscillatory, incompressible flow in a circular tube. Numerical investigations by several authors have shown that the flow is stable for infinitesimal, axisymmetric perturbations. The goal of our analysis is to study the hydrodynamic stability of the flow by solving for the stream function of axisymmetric perturbances, which satisfies the fourth order Orr-Sommerfeld equation. We present a semi-analytical approach based on the Galerkin projection of the Orr-Sommerfeld equation onto an approximation functional space solution, spanned by a finite set of the eigenfunctions of the longwave limit Orr-Sommerfeld operator. Convergence to the exact solution is obtained as the number of eigenfunctions approaches infinity. The truncated Floquet system is solved using the Runge-Kutta method and results confirm the known stability characteristics of the flow.

Key words: stability, pulsatile flow, Orr-Sommerfeld operator, Galerkin projection, eigenfunction.

1 Introduction

The study of pulsatile tube flow appears to have been first considered in the context of arterial hemodynamics in the mid-1950s. Womersley and co-workers obtained an exact solution of the Navier-Stokes equations for the fully-developed velocity profile of a oscillatory, incompressible flow in a circular tube ([1],[2]). The Womersley problem has recently found renewed significance in its application to MEMS microfluidic engineering applications. A common feature of many of the microfluidic devices described in the literature that incorporate micro-scale pumping is that the flow is a pulsatile one ([3],[4]); the stability of pulsatile flow is therefore relevant for MEMS pump operations. Some authors have also examined pulsatile microchannel flows as potential laminar mixing strategies [5]. Prior numerical investigations of the stability of pulsatile pipe flow have been performed by Tozzi and Kerczek ([6]) using Galerkin projection with Tchebycheff basis. They found that the flow is slightly more stable than the steady Hagen-Poiseuille flow, which is stable for infinitesimal disturbances for all the values of Reynolds number ([7],[8]). An excellent summary of the stability of time-periodic flows is found in Davis [9].

In this paper we revisit the linear stability of pulsatile flow for infinitesimal axisymmetric disturbances. The evolution of the perturbations in time is investigated on the basis of the Floquet theory ([10],[11]). The Orr-Sommerfeld equation governing the perturbation stream function of the disturbances [12] is solved by a Galerkin projection method [13]. A set of 'natural' eigenfunctions corresponding to the longwave limit Orr-Sommerfeld operator are derived and used as a complete, orthonormal basis for the expansion. Our results confirm the previously reported results that the pulsatile flow is linearly stable.

1.1 The Orr-Sommerfeld Equation for Perturbed, Oscillatory Pipe Flow

Consider the pulsatile flow dynamics in a pipe of circular cross section of radius R driven by an imposed pressure gradient. The streamwise velocity $W(r,t)$ satisfies the following initial boundary value problem

$$\frac{\partial W}{\partial t}-\upsilon\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial W}{\partial r}\right)=-\frac{1}{\rho}\frac{\partial P}{\partial z},\qquad \frac{\partial P}{\partial z}=-\left[P_0+K_\omega\exp(\omega t)\right] \tag{1}$$

with the no-slip condition at the boundary of the pipe, and boundedness of the velocity field at the centerline of the tube. The solution for the radial velocity profile $W(r,t)$ is given by ([1],[2])

$$W(r,t)=\frac{P_0}{4\mu}\left(R^2-r^2\right)+\frac{R^2}{\mu Wo^2 i}\left[1-\frac{J_0\left(i^{\frac{3}{2}}Wo\frac{r}{R}\right)}{J_0\left(i^{\frac{3}{2}}Wo\right)}\right]K_\omega\exp\left(i\omega t\right) \tag{2}$$

where R is the tube radius, J_0 is the Bessel function of the first kind of order zero, μ is the viscosity, and the parameter Wo has become known as the Womersley number. The Womersley number is defined by $Wo=\sqrt{\rho\omega R^2/\mu}$ and represents the ratio of oscillatory inertia to viscous forces; it may also be interpreted as a Reynolds number for the flow using ωR as the velocity scale. In order to study the stability of the flow an axisymmetric perturbance is superimposed to the basic flow. Since the perturbed flow field is bidimensional, it can be expressed by means of a stream function of the form $\Psi(r,z,t)=\psi(r,t)e^{i\alpha z}$ where α is the streamwise wave number. The radial and streamwise velocity components of the perturbance are then expressed in terms of Ψ as

$$u=-\frac{1}{r}\frac{\partial\Psi}{\partial z}=-\frac{\psi(r,t)}{r}i\alpha e^{i\alpha z}\qquad w=\frac{1}{r}\frac{\partial\Psi}{\partial r}=\frac{1}{r}\frac{\partial\psi}{\partial r}e^{i\alpha z} \tag{3}$$

and the condition of incompressibility is automatically satisfied. Substitution of the basic flow (2) and the perturbation velocity (3) into the general Navier-Stokes equations yields, after neglecting nonlinear terms, the following Orr-Sommerfeld equation for the stream function ψ (for details, see [7])

$$\mathcal{L}\psi_t-Wi\alpha^3\psi+i\alpha\left(-\psi\mathcal{L}W+W\mathcal{L}\psi\right)=\upsilon\mathcal{L}^2\psi. \tag{4}$$

Here the differential operator is defined by $\mathcal{L}=\partial^2/\partial r^2-r^{-1}\partial/\partial r-\alpha^2$. The no-slip condition at the wall of the pipe and the boundness of the flow at the centerline impose the boundary conditions

$$\psi(r,t)=\frac{\partial\psi}{\partial r}(r,t)=0\quad at\ \ r=0,R \tag{5}$$

for the stream function ψ.

It is convenient to non-dimensionalize the Orr-Sommerfeld equation by defining the following change of variables

$$W^*=\frac{W}{U_0}\quad U^*=\frac{U}{U_0}\quad t^*=\frac{t}{T}\quad r^*=\frac{r}{R}\quad z^*=\frac{z}{L}\quad \psi^*=\frac{\psi}{U_0R^2}$$

$$\alpha^*=\alpha R\quad Re=\frac{U_0R}{\upsilon}\quad T=\frac{R}{U_0}\quad \omega^*=\omega T=\frac{Wo^2}{Re}$$

where U_0 is a typical velocity, T is a convective velocity and ω^* is the Strouhal number. In terms of these variables, the Orr-Sommerfeld equation (4) transforms to

$$\begin{cases} \mathcal{L}\psi_t - Wi\alpha^3\psi + i\alpha\left(-\psi\mathcal{L}W + W\mathcal{L}\psi\right) = Re^{-1}\mathcal{L}^2\psi \\ \\ \psi(r,t) = \frac{\partial\psi}{\partial r}(r,t) = 0 \quad at \;\; r = 0, 1. \end{cases} \tag{6}$$

where we shall hereafter omit the asterisks. For simplicity, we now consider the case of a pulsatile flow containing a single harmonic which has the form $W = W_0 + W_1 \exp i\omega t$ where

$$\begin{aligned} W_0 &= v\left(1 - r^2\right) \qquad W_1 = \Lambda\frac{1}{i}\left[1 - \frac{J_0\left(i^{3/2}Wo\, r\right)}{J_0\left(i^{3/2}Wo\right)}\right] \\ \Lambda &= \frac{K_\omega R}{\rho\omega\, U_0^2} \quad v = \frac{P_0}{4\mu R^2 U_0} \end{aligned}$$

To further facilitate the analysis of the pulsatile effects on the steady flow we choose $U_0 = P_0/4\mu R^2$ and take $\omega = 1$. In this case, we have that $Wo \sim \sqrt{Re}$ and flow pulsatility will generally be significant for all but very low values of the Reynolds number.

2 Galerkin projection of the Orr-Sommerfeld equation

2.1 A complete set of eigenfunctions for the Orr-Sommerfeld equation

Exact solutions of the Orr-Sommerfeld equation (6) are difficult to obtain. A simple solution can be obtained for the special case of $\alpha = 0$ for which Eq. (6) reduces to

$$\begin{cases} \tilde{\mathcal{L}}\tilde{\psi}_t = Re^{-1}\tilde{\mathcal{L}}^2\tilde{\psi} \\ \\ \tilde{\psi}(r,t) = \frac{\partial\tilde{\psi}}{\partial r}(r,t) = 0 \quad at \;\; r = 0, 1 \end{cases} \tag{7}$$

where $\tilde{\mathcal{L}} = r\partial/\partial r\left(r^{-1}\partial/\partial r\right)$ is a reduced operator. Eq. (7) admits the following eigenfunction series expansion solution

$$\tilde{\psi}(r,t) = \sum_n a_n\phi_n(r)\exp(\lambda_n t)$$

where the set of coefficients $\{a_n\}$ defines the initial conditions and the set of eigenfunctions $\{\phi_n(r)\}$ satisfies the eigenvalue problem

$$\begin{cases} \tilde{\mathcal{L}}^2\phi_n = Re\lambda_n\tilde{\mathcal{L}}\phi_n \\ \\ \phi_n = \frac{\partial\phi_n}{\partial r} = 0 \quad \text{at } r = 0 \text{ and } r = 1. \end{cases}$$

One readily finds

$$\phi_n(r) = \frac{\sqrt{2}}{y_n}r\left(r - \frac{J_1\left(y_n r\right)}{J_1\left(y_n\right)}\right) \qquad \lambda_n = -\frac{y_n^2}{R_e} \text{ eigenvalues}$$

where $J_1(x)$ and $J_2(x)$ are Bessel functions of first kind of order 1 and 2, respectively and y_n are the roots of J_2. The set of eigenfunctions $\{\phi_n\}$ can be shown to form a set of complete orthonormal basis for the functional space $\mathcal{F}\left([0,1]\right)$ defined by

$$\mathcal{F}\left([0,1]\right) = \left\{f \in L_2\left([0,1]\right) \;:\, f = \frac{\partial f}{\partial r} = 0 \;\; at \;\; r = 0, 1\right\}$$

where $L_2([0,1])$ is the Lebesque space, with respect to the following scalar product

$$\langle f, g\rangle = -\frac{1}{2}\int_0^1 \frac{1}{r}\, f\, \tilde{\mathcal{L}} g\, dr = \frac{1}{2}\int_0^1 \frac{1}{r}\frac{\partial f}{\partial r}\frac{\partial g}{\partial r} dr. \tag{8}$$

We shall use the completeness of the set $\{\phi_n\}$, to solve for the Orr-Sommerfeld equation (4) when $\alpha \neq 0$ by means of a Galerkin projection.

2.2 Galerkin projection

We seek an expansion for the unknown stream function ψ as

$$\psi(r,t) = \sum_{n=1}^{\infty} A_n(t)\phi_n(r) \tag{9}$$

where $A_n(t)$ are time-dependent unknown coefficients. Under this expansion the boundary conditions for ψ are automatically satisfied. We now project the Orr-Sommerfeld equation (4) onto the space solution $\mathcal{F}([0,1])$ using the scalar product defined in Eq. (8)

$$\int_0^1 -\left(\mathcal{L}\psi_t - Wi\alpha^3\psi + i\alpha\left(-\psi\mathcal{L}W + W\mathcal{L}\psi\right) - \frac{1}{R_e}\mathcal{L}^2\psi\right)\frac{1}{r}\tilde{\mathcal{L}}\phi_k\, dr = 0 \qquad k = 1, 2, ... \tag{10}$$

Substituting the expansion (9) for ψ into (10) and using the orthogonality of the ϕ_n, one obtains an infinite system of ordinary differential equations for the set of coefficients $\{A_n\}$. In matrix form, the system is given by

$$\frac{d\mathbf{a}}{dt} = \left[\mathbf{M}_0 + \mathbf{M}_1 e^{i\omega t}\right]\ \mathbf{a} \tag{11}$$

where we have defined the following vectors and matrices as

$$(\mathbf{a})_n = A_n$$

$$(\mathbf{M}_0)_{nk} = i\left(-\alpha^3 P_{nk} + \alpha H_{nk}\right)\left(y_n^2 + \alpha^2\right)^{-1} + Re^{-1}\left(y_n^2 + 2\alpha^2 + \tfrac{\alpha^4}{y_n^2}\right) y_n^2\left(y_n^2 + \alpha^2\right)^{-1}\delta_{nk}$$

$$(\mathbf{M}_1)_{nk} = i\left(-\alpha^3 Q_{nk} + \alpha G_{nk}\right)\left(y_n^2 + \alpha^2\right)^{-1}$$

where δ_{nk} is the Kronecker delta and

$$P_{nk} = -\int_0^1 \frac{1}{r} W_0 \tilde{\mathcal{L}}\phi_k \phi_n dr \qquad Q_{nk} = -\int_0^1 \frac{1}{r} W_1 \tilde{\mathcal{L}}\phi_k \phi_n dr$$

$$H_{nk} = -\int_0^1 \frac{1}{r} W_0 \tilde{\mathcal{L}}\phi_k \tilde{\mathcal{L}}\phi_n dr \qquad G_{nk} = -\int_0^1 \frac{1}{r}\tilde{\mathcal{L}}\phi_k\left(-\phi_n\tilde{\mathcal{L}}W_1 + W_1\tilde{\mathcal{L}}\phi_n\right) dr$$

Because the basic flow is periodic we expect that the perturbations will mantain the same temporal character but with exponential decay/growth in time. According to Floquet theory [10], the solution of the infinite system (11) is of the form

$$\mathbf{a}(t) = \exp\beta t\ \mathbf{g}(t)$$

where β is the Floquet exponent and $\mathbf{g}(t)$ is $\mathcal{T}-$periodic vector. The Floquet exponent is related to the eigenvalues μ of the the fundamental matrix $\mathbf{G}(t)$ of the system (11) evaluated at time $t = \mathcal{T}$ as $\beta = \ln\mu/\mathcal{T}$. The fundamental matrix $\mathbf{G}$ is evaluated by means of a Runge-Kutta method.

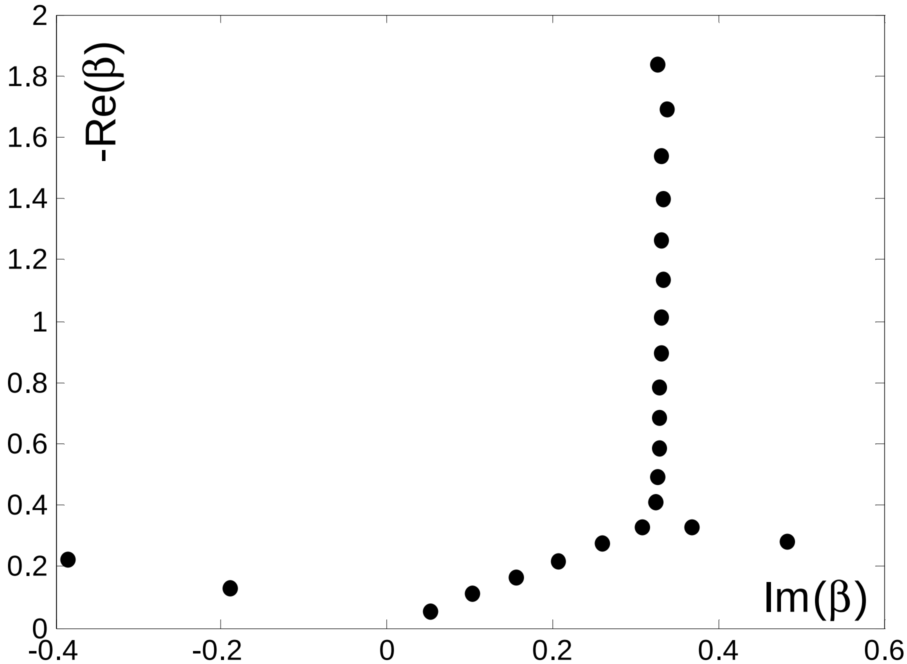

Figure 1: The Floquet exponents β for a basic flow state with Re $= 3000$, $Wo = 54.8$, $\omega = 1$, and a perturbation with wavenumber $\alpha = 1$. An eigenfunction expansion of $N = 40$ terms has been computed.

3 An Illustrative Example

Finally, we demonstrate the use of our eigenfunction expansion method in the examination of an axisymmetrically-perturbed pulsatile pipe flow. The basic flow state is characterized by a Reynolds number Re $= 3000$, $\omega = 1$, and $Wo = 54.8$; we assume an axial perturbation with wavenumber $\alpha = 1$. An eigenfunction expansion is obtained consisting of $N = 40$ terms. The Floquet exponents of the truncated system (11) are plotted in Fig.1 in the complex plane. The eigenvalues with imaginary part close to zero, called 'wall modes', are dumped indicating that the pulsatile flow is linearly stable.

4 Conclusions

The linear stability of pulsatile flow has been revisited for infinitesimal axisymmetric disturbances. The Orr-Sommerfeld equation governing the perturbation stream function of the disturbances has been solved by means of a Galerkin projection method where the complete orthonormal eigenfunctions of the longwave limit Orr-Sommerfeld operator have been chosen as basis for the expansion. Our results confirm that the pulsatile flow is linearly stable.

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