Three-Dimensional FEM Sensitivity for Biomedical Optical Tomography using Adjoint Methods

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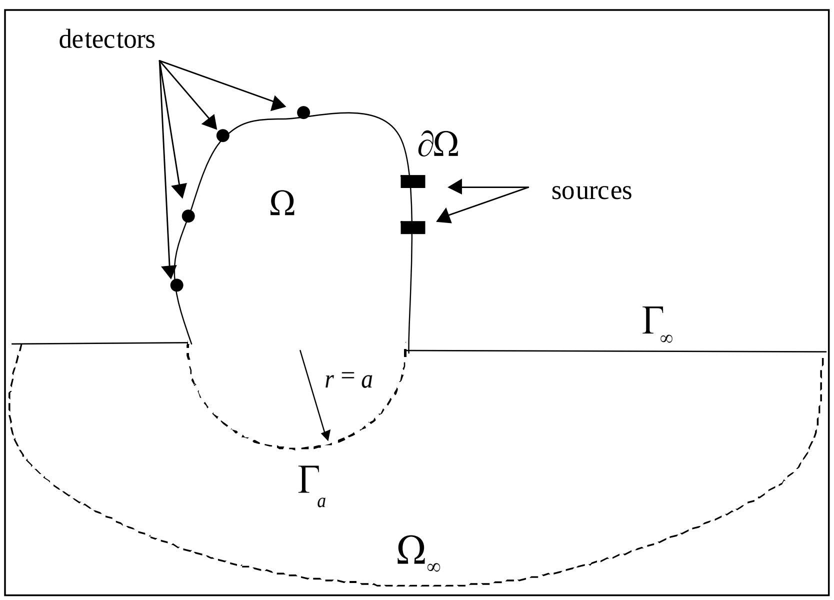
FREQUENCY-DOMAIN PHOTON MIGRATION PDE's

$$\begin{cases} -\nabla \bullet \left(D_x \nabla \Phi_x\right) + k_x \Phi_x = S \\ -\nabla \bullet \left(D_m \nabla \Phi_m\right) + k_m \Phi_m = \beta \Phi_x \end{cases}$$

$$\begin{cases} D_x \dfrac{\partial \Phi_x}{\partial n} + r_x \Phi_x = 0 \\ D_m \dfrac{\partial \Phi_m}{\partial n} + r_m \Phi_m = 0 \end{cases} \quad on \quad \partial\Omega$$



$$D_x = \frac{1}{3(\mu_{axi} + \mu_{axf} + \mu_{sx})} \qquad D_m = \frac{1}{3(\alpha_1 \mu_{axi} + \alpha_2 \mu_{axf} + \alpha_3 \mu_{sx})}$$

$$k_x = \frac{i\omega}{c_x} + \mu_{axi} + \mu_{axf} \qquad k_m = \frac{i\omega}{c_m} + \alpha_1 \mu_{axi} + \alpha_2 \mu_{axf}$$

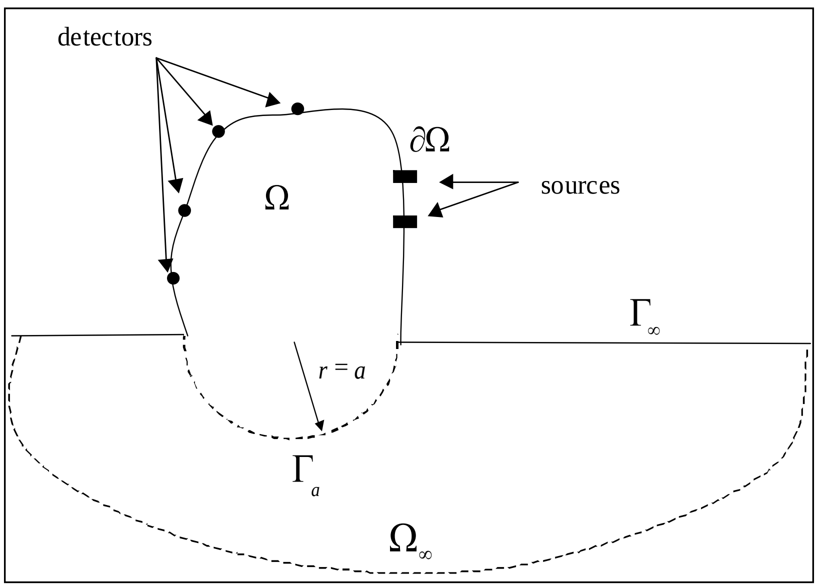
$$\beta = \phi \frac{1 - i\omega\tau}{1 + (\omega\tau)^2} \mu_{axf}$$

THE INVERSE PROBLEM

$$L\Phi(x,\mu) = f(\mu) \quad L \;\; diffusion - decay \;\; operator$$

$$F(\mu) = \sum_{n=1}^{N}\left(\Phi_n^{\det} - \Phi_{meas}^{\det}\right)^2$$

$$find \;\; \mu \quad such \;\; that \quad F(\mu) \;\; \min$$



Sensitivity

$$\frac{\delta F}{\delta \mu} = 2\sum_{n=1}^{N}\left(\Phi_n^{\det} - \Phi_{meas}^{\det}\right)\frac{\delta \Phi_n^{\det}}{\delta \mu} = 0$$

$$\sum_{n=1}^{N}\left(\overline{\Phi}_n^{\det} + \frac{\delta \Phi_n^{\det}}{\delta \mu}\delta\mu - \Phi_{meas}^{\det}\right)\frac{\delta \Phi_n^{\det}}{\delta \mu} = 0$$

THE FINITE DIFFERENCE APPROACH

$$\boxed{\frac{\delta\Phi}{\delta\mu} = \frac{\Phi(x,\mu+\delta\mu) - \Phi(x,\mu)}{\delta\mu} + o(\delta\mu)}$$

$$\Phi(x,\mu) \quad satisfies \quad L\Phi(x,\mu) = f(\mu)$$

$$\Phi(x,\mu+\delta\mu) \quad satisfies \quad L\Phi(x,\mu+\delta\mu) = f(\mu+\delta\mu)$$

We need to solve two PDEs over all the domain

in order to evaluate $\left(\frac{\delta\Phi}{\delta\mu}\right)^{\mathrm{det}}$ only at the location of the detectors !!!!

Computationally demanding !!!!

$$\frac{\delta\Phi}{\delta\mu} \quad ? \qquad\qquad f\left(\Phi,\mu\right)=0$$

$$\mu \;\rightarrow\; \mu+\delta\mu \;\rightarrow\; \Phi+\delta\Phi \;\Rightarrow\; f\left(\Phi+\delta\Phi,\mu+\delta\mu\right)=0$$

$$f\left(\Phi,\mu\right)+\frac{\partial f}{\partial\Phi}\delta\Phi+\frac{\partial f}{\partial\mu}\delta\mu+o(\delta\Phi,\delta\mu)=0$$

$$\frac{\partial f}{\partial\Phi}\delta\Phi=-\frac{\partial f}{\partial\mu}\delta\mu$$

Perturbational equation

$$\delta\Phi=\left(\frac{\partial f}{\partial\Phi}\right)^{-1}\left(-\frac{\partial f}{\partial\mu}\delta\mu\right)$$

Inverse function

$$f(\Phi,\mu)=0 \quad \Rightarrow \quad L\Phi(x,\mu)=S$$

$$\boxed{L=-\nabla\bullet\left(D(\mu)\nabla\Phi\right)+k(\mu)\Phi}$$

$$\mu \;\rightarrow\; \mu+\delta\mu \;\rightarrow\; \Phi+\delta\Phi \qquad \begin{cases} (L+\delta L)(\Phi+\delta\Phi)=S & on\ \Omega \\ (D+\delta D)\dfrac{\partial(\Phi+\delta\Phi)}{\partial n}+r(\Phi+\delta\Phi)=0 & on\ \partial\Omega \end{cases}$$

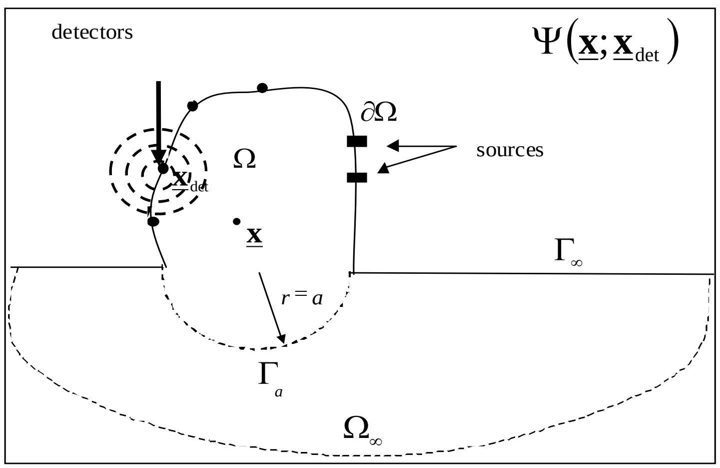
Perturbational equations

$$\begin{cases} L(\delta\Phi)=G(\Phi,\mu,\delta\mu) & on\ \ \Omega \\ D\dfrac{\partial\delta\Phi}{\partial n}+r\delta\Phi=j(\Phi,\mu,\delta\mu) & on\ \ \partial\Omega \end{cases}$$

THE GREEN FUNCTION

$$\begin{cases} L(\delta\Phi) = G(\Phi, \mu, \delta\mu) & on \;\; \Omega \\ D\dfrac{\partial\delta\Phi}{\partial n} + r\delta\Phi = j(\Phi, \mu, \delta\mu) & on \;\; \partial\Omega \end{cases}$$

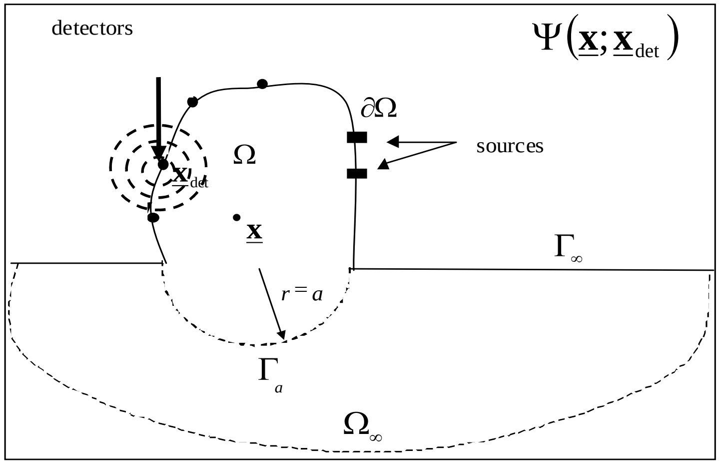
$$\delta\Phi = \mathscr{F}\Big(G\big(\Phi, \mu, \delta\mu\big), j\big(\Phi, \mu, \delta\mu\big)\Big) \;\; ???$$

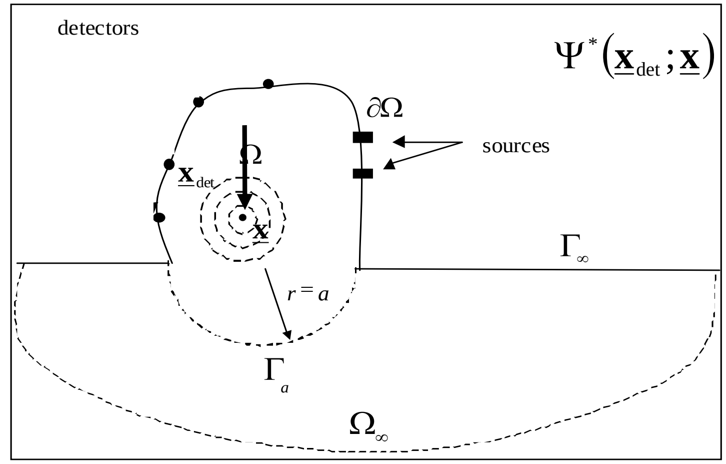


$$\begin{cases} L^*(\Psi) = \delta(\underline{\mathbf{x}} - \underline{\mathbf{x}}_{\det}) & on \;\; \Omega \\ D\dfrac{\partial\Psi}{\partial n} + r\Psi = 0 & on \;\; \partial\Omega \end{cases}$$

$$\delta\Phi(\underline{\mathbf{x}}_{\det}) = \int_\Omega \Psi(\underline{\mathbf{x}}; \underline{\mathbf{x}}_{\det}) G(\Phi, \mu, \delta\mu) d\Omega + \int_{\partial\Omega} \Psi(\underline{\mathbf{x}}; \underline{\mathbf{x}}_{\det}) j(\Phi, \mu, \delta\mu) dS$$

THE FUNDAMENTAL SOLUTION



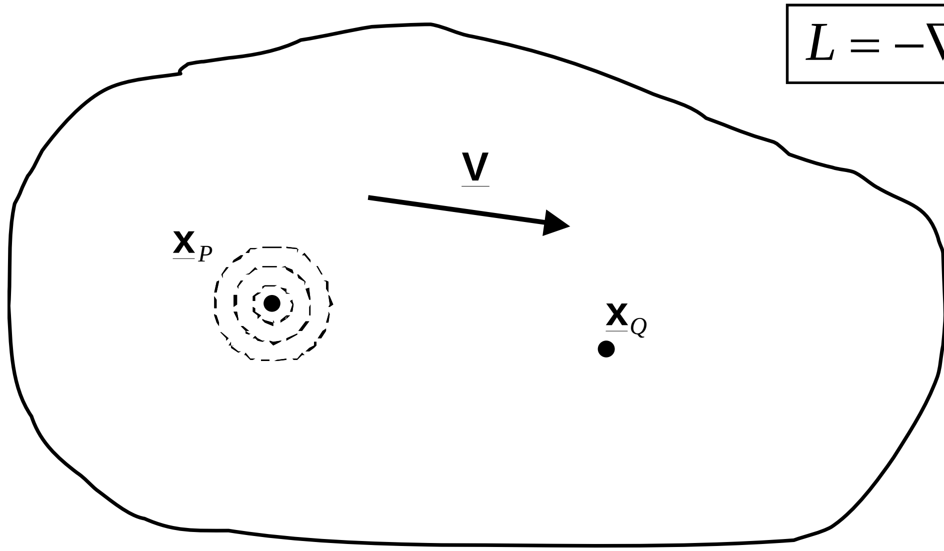


$$\delta\Phi(\underline{\mathbf{x}}_{\text{det}}) = \int_{\Omega} \Psi^*(\underline{\mathbf{x}}_{\text{det}};\underline{\mathbf{x}}) G(\Phi,\mu,\delta\mu)\, d\Omega + \int_{\partial\Omega} \Psi^*(\underline{\mathbf{x}}_{\text{det}};\underline{\mathbf{x}})\, j(\Phi,\mu,\delta\mu)\, dS$$

$$\begin{cases} L(\delta\Phi) = G(\Phi,\mu,\delta\mu) & on\ \ \Omega \\ D\dfrac{\partial\delta\Phi}{\partial n} + r\delta\Phi = j(\Phi,\mu,\delta\mu) & on\ \ \partial\Omega \end{cases}$$

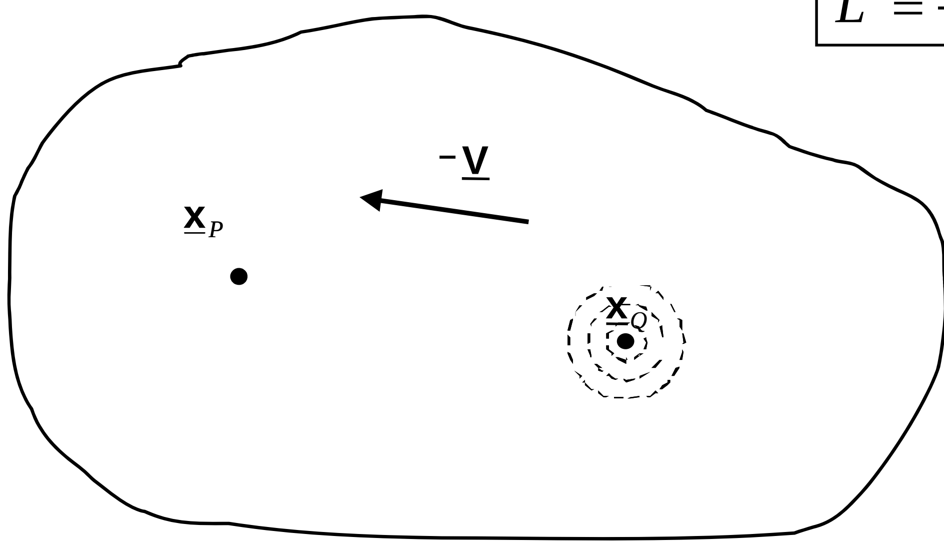
Fundamental solution $\Psi^*\left(\underline{\mathbf{x}}_Q, \underline{\mathbf{x}}_P\right)$

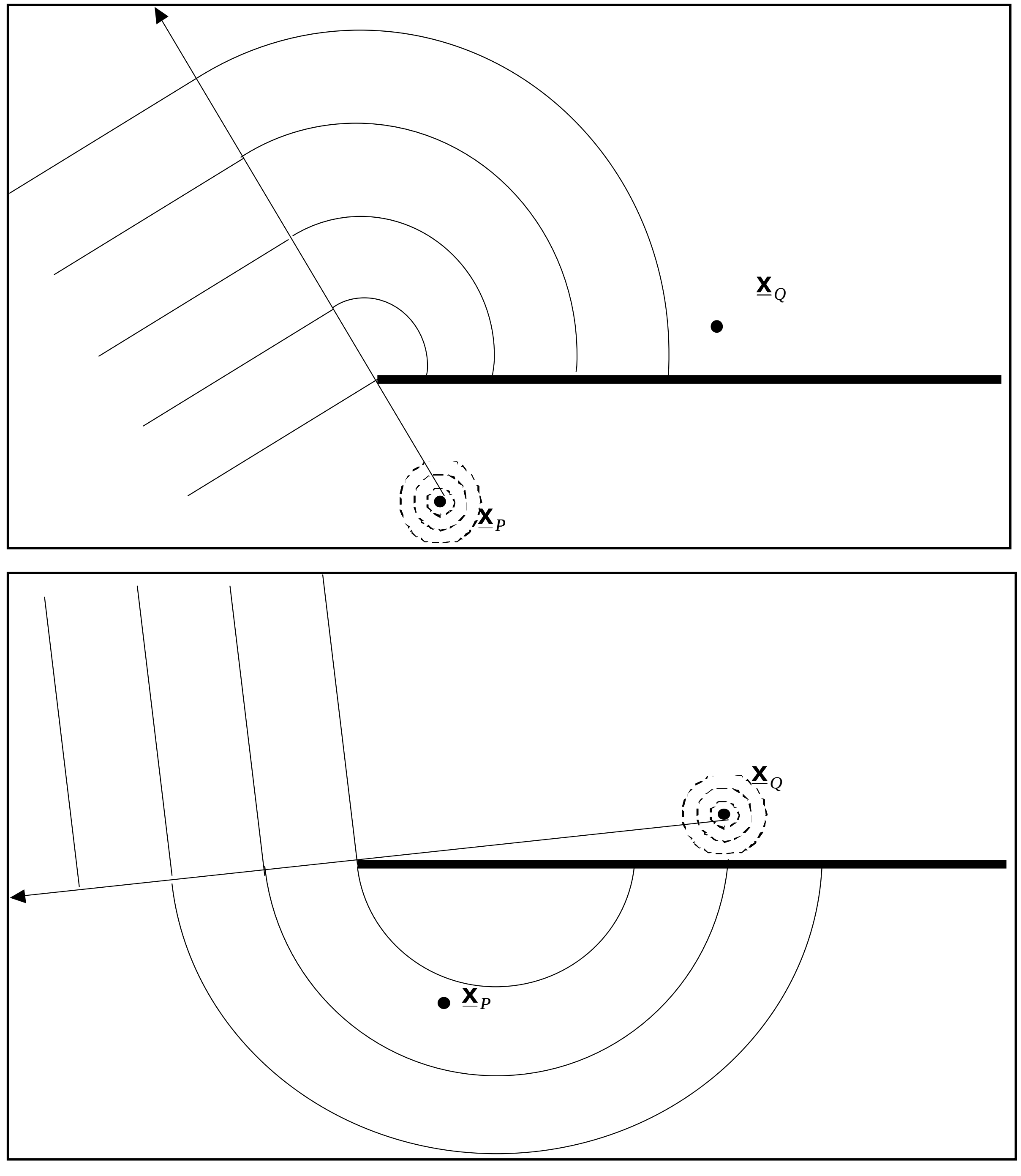
$$L = -\nabla \bullet \left(D \nabla \Phi\right) + \underline{V} \bullet \nabla \Phi$$



Adjoint solution (Green function)

$$L^* = -\nabla \bullet \left(D \nabla \Phi\right) - \underline{V} \bullet \nabla \Phi$$





$\underline{\mathbf{x}}_Q$
$\underline{\mathbf{x}}_P$
$\underline{\mathbf{x}}_Q$
$\underline{\mathbf{x}}_P$