Quasi-deterministic wave structures in stochastic gaussian fields
A paradigm for inverse problems ?

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GAUSSIAN SEA STATES

ERGODICITY

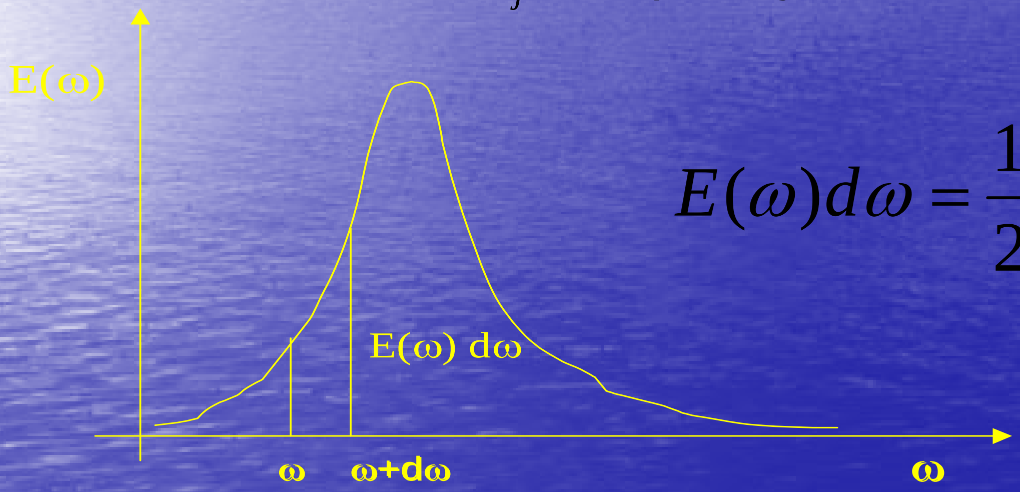
QUASI-DETERMINISTIC WAVE STRUCTURES

INVERSE PROBLEMS

GAUSSIAN SEA STATES

$$\eta(t) = \sum_{j=1}^{N} a_j \cos(\omega_j t + \varepsilon_j)$$

$$\varepsilon_j \quad uniformly\ random\ in\ [0, 2\pi]$$

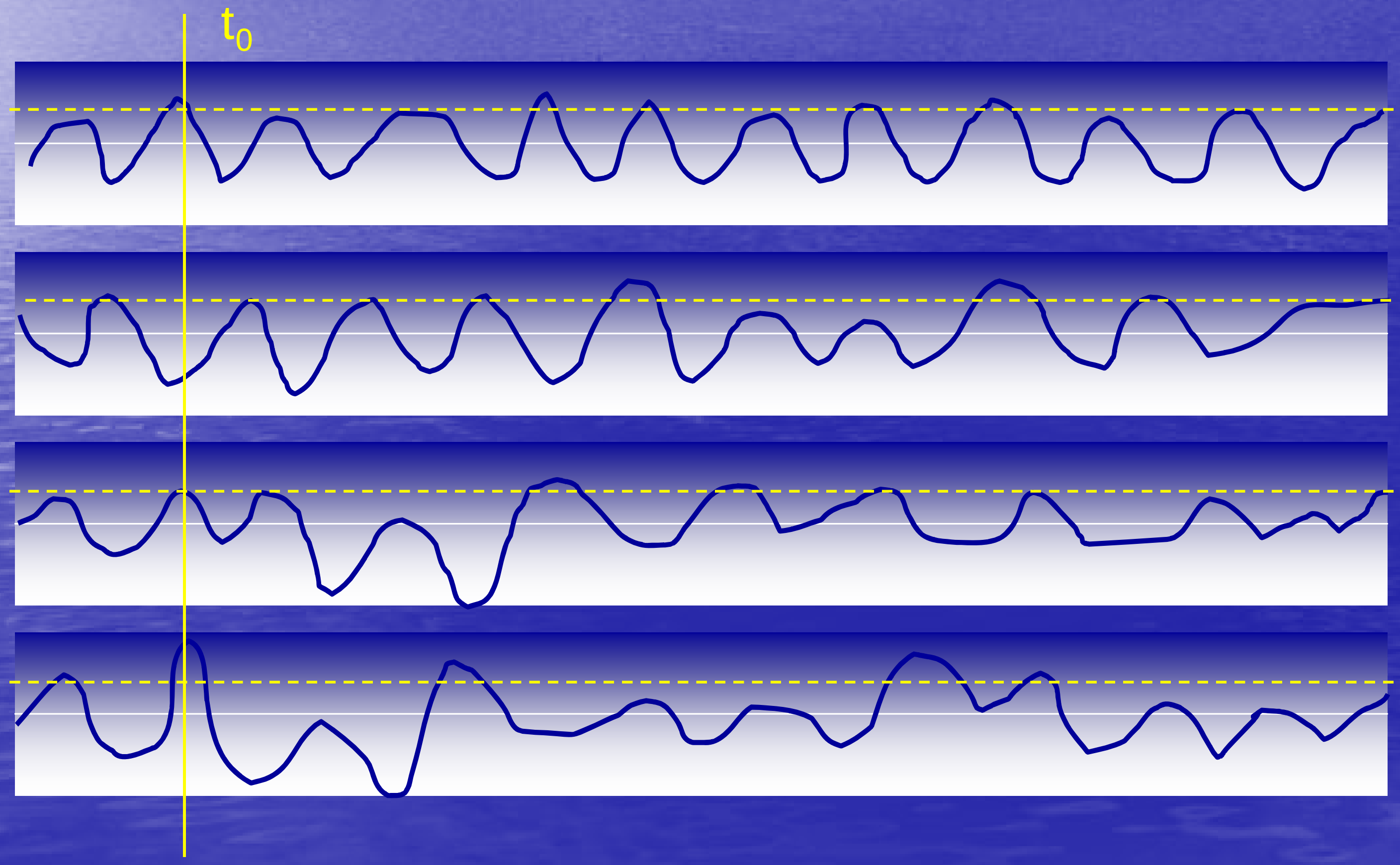


$$E(\omega)d\omega = \frac{1}{2}\sum_j a_j^2 \qquad \omega < \omega_j < \omega + d\omega$$

Stationarity Ergodicity Gaussianity

$$\Pr\left[\eta(t_0) > z\right] = \frac{\#\ realizations\ in\ which\ \eta\ \ is\ greater\ than\ z\ \ at\ the\ time\ t_0}{\#\ realizations}$$

$$\Pr\left[\eta(t) > z\right] = \frac{time\ during\ which\ the\ wave\ elevation\ \eta\ is\ greater\ than\ z}{total\ time\ of\ one\ realization}$$



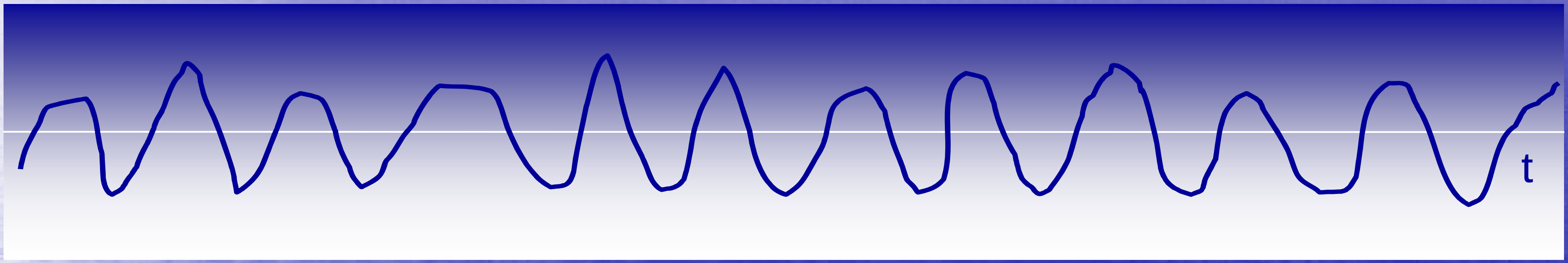
A STATIONARY GAUSSIAN NON ERGODIC PROCESS

$$\eta(t) = h$$

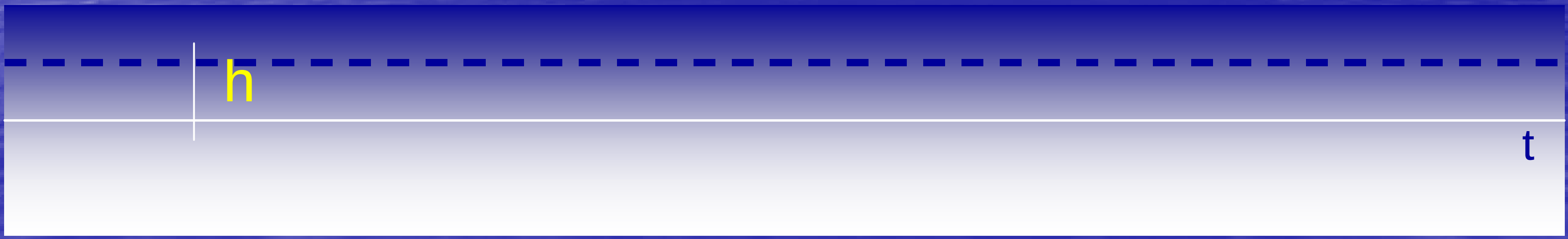
h constant gaussian

ERGODIC THEOREM

$$\eta(t) = \sum_{j=1}^{N} a_j \cos(\omega_j t + \varepsilon_j) \qquad \overline{\eta} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \eta(\tau)\, d\tau$$



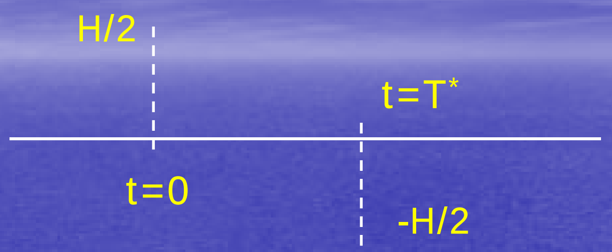
$$\eta(t) = h \qquad \overline{\eta} \neq \lim_{T\to\infty} \frac{1}{T}\int_0^T \eta(\tau)\, d\tau = h$$



QUASI-DETERMINISTIC WAVE STRUCTURES

Let us assume we know that

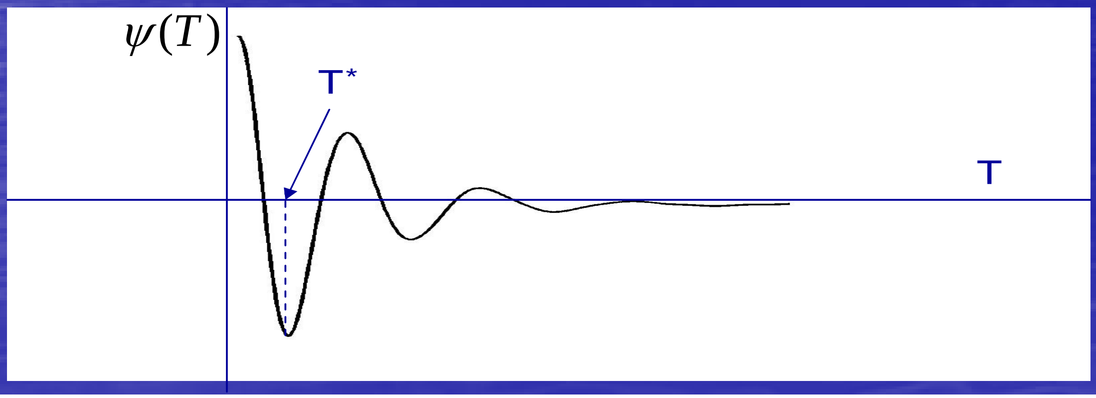
$$\eta(0) = \frac{H}{2} \qquad \eta(T^*) = -\frac{H}{2}$$



What is the probability that

$$\eta(t) \in [\eta, \eta + d\eta] \quad ?$$

$$\Pr\left[\eta(t) = \eta / \eta(0) = \frac{H}{2}, \eta(T^*) = -\frac{H}{2}\right]$$



CONDITIONAL PROBABILITY

$$\Pr\left[\eta(t)=\eta/\eta(0)=\frac{H}{2},\eta\left(T^{*}\right)=-\frac{H}{2}\right]$$

⇓

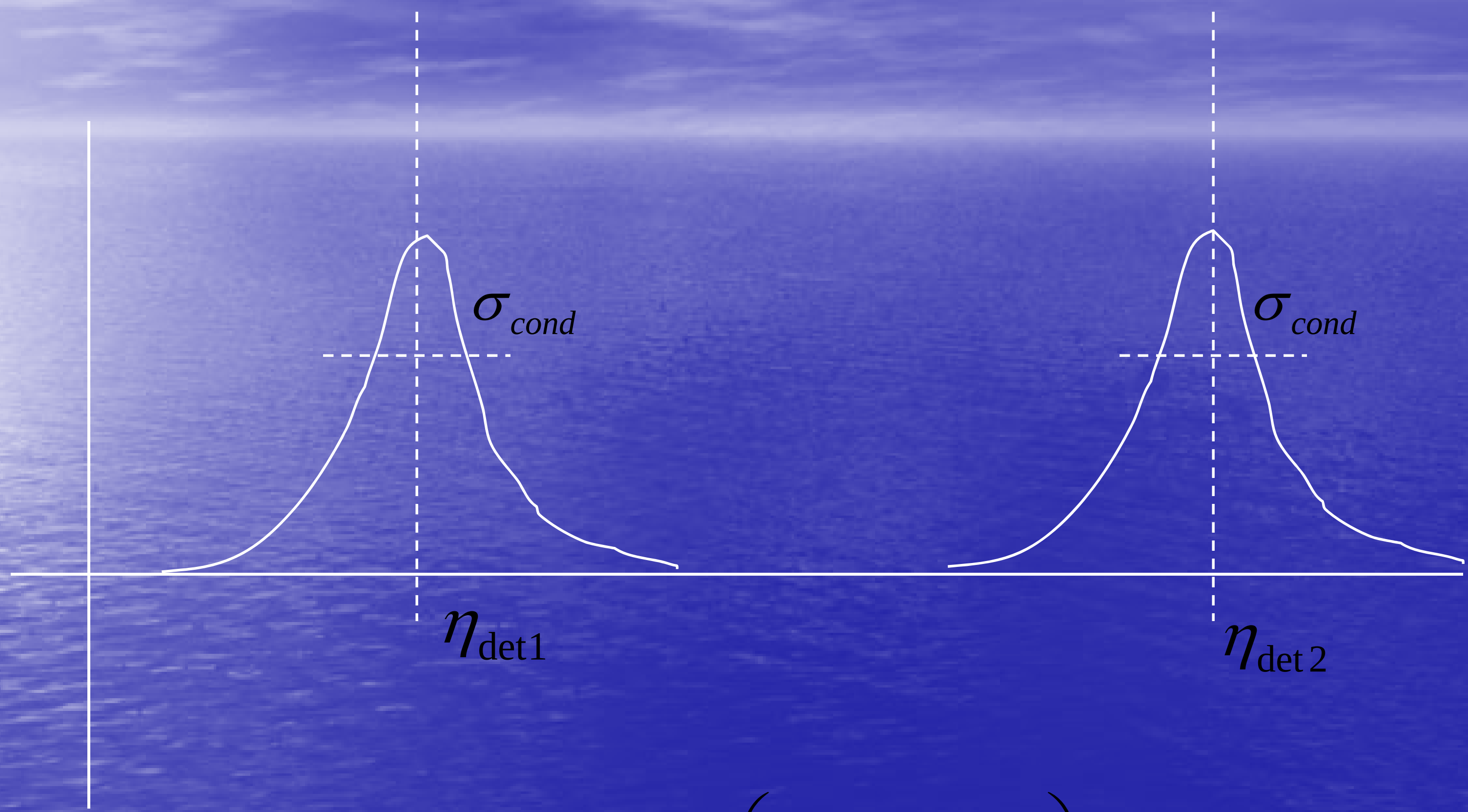
$$\frac{1}{\sqrt{2\pi\sigma_{cond}^{2}}}\exp\left[-\frac{1}{2\sigma_{cond}^{2}}\left(\eta-\eta_{\det}\right)^{2}\right]$$

$$\eta_{\det}(t)=\frac{H}{2}\frac{\psi(t)-\psi\left(t-T^{*}\right)}{\psi(0)-\psi\left(T^{*}\right)}$$

$$\sigma_{cond}\leq\sigma$$

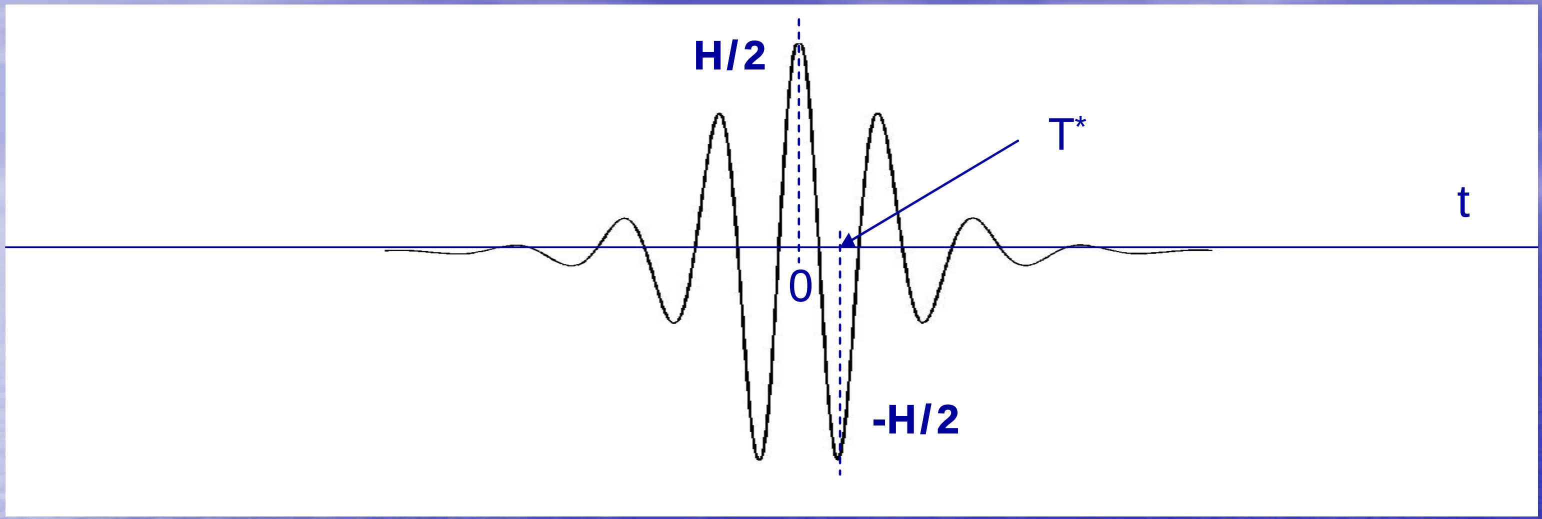
$$\frac{\eta_{\det}(t)}{\sigma_{cond}}\geq\frac{\eta_{\det}(t)}{\sigma}=k\frac{H}{\sigma}\rightarrow\infty \quad if \quad \frac{H}{\sigma}\rightarrow\infty$$

PROBABILITY "FREEZING"

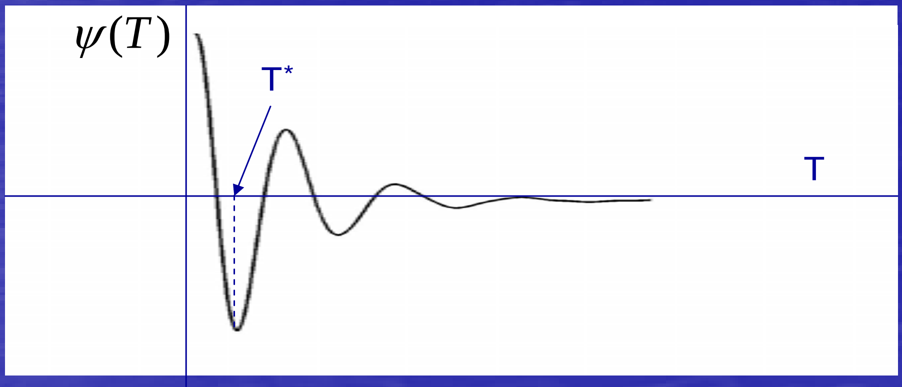


$$\eta \in \eta_{\det}\left(1 \pm 3\frac{\sigma_{cond}}{\eta_{\det}}\right)$$

THE HIGH WAVE PROFILE



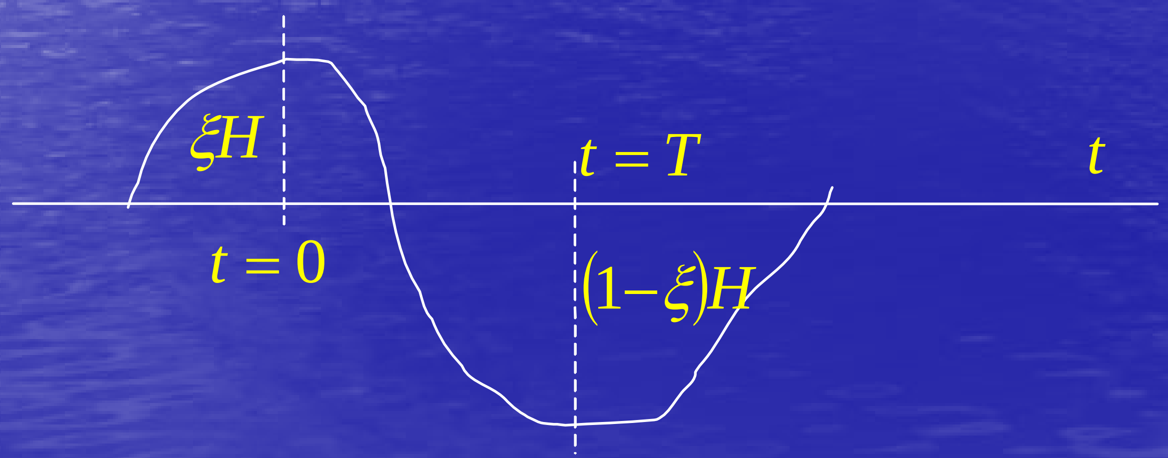
$$\eta_{\det}(t) = \frac{H}{2}\frac{\psi(t) - \psi(t - T^*)}{\psi(0) - \psi(T^*)}$$



NECESSARY CONDITIONS TO HAVE A WAVE IN A GAUSSIAN SEA STATE

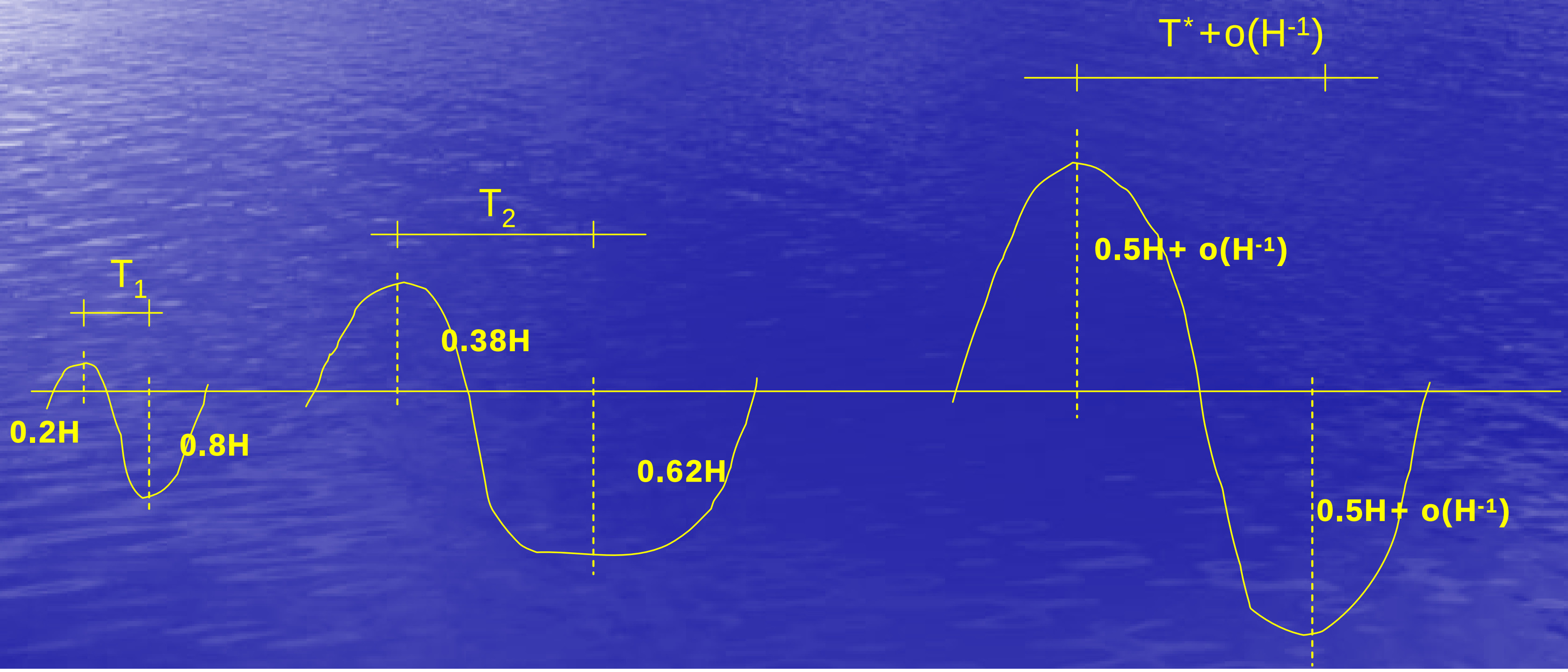
$$\eta(0) = \frac{H}{2} \qquad \eta(T^*) = -\frac{H}{2} \qquad \frac{H}{\sigma} \to \infty$$

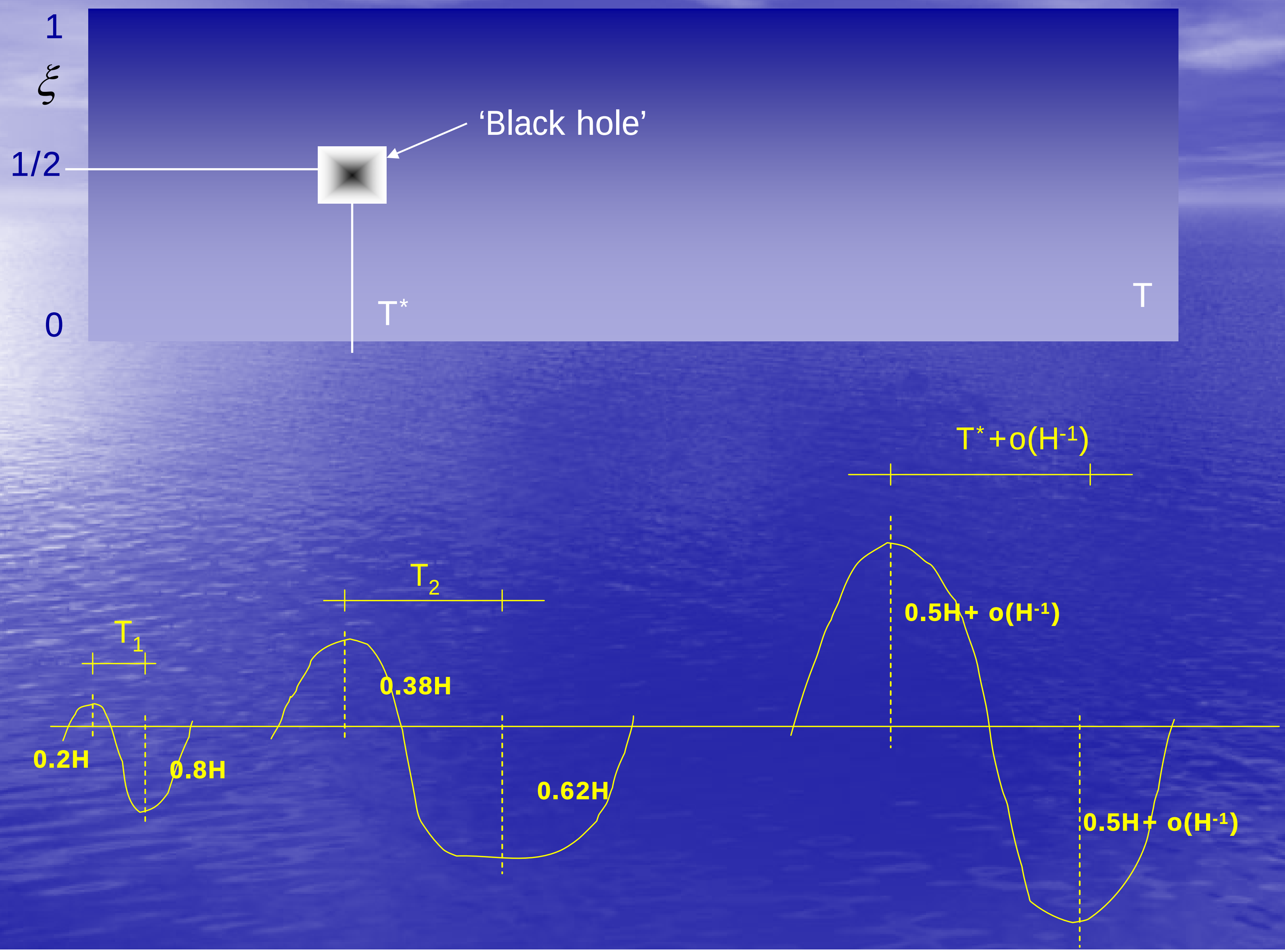
$$\Pr\left[\eta(0) = \xi H + d\eta_1 \,,\ \eta(T) = (1-\xi)H + d\eta_2\right] \qquad ?$$



$$P(H,\xi,T) \propto \exp\left[-\frac{1}{2}\left(\frac{\sigma^2}{\sigma^2-\psi(T)}+\beta\left(\xi-\frac{1}{2}\right)^2\right)\left(\frac{H}{\sigma}\right)^2\right]$$

$$\frac{P(H,\xi,T)}{P\left(H,\frac{1}{2},T^*\right)} \to 0 \quad as \quad \frac{H}{\sigma} \to \infty$$





1
ξ
1/2
0
'Black hole'
T*
T
T* + o(H-1)
T2
T1
0.5H+ o(H-1)
0.38H
0.2H
0.8H
0.62H
0.5H+ o(H-1)

INVERSE PROBLEMS

$$\Gamma_{\mu}(u) = S$$

$$find \quad \mu \quad such \quad that \; u = u_0 \quad on \;\; \partial\Omega$$

By means of the Green function

$$u_0(x,t)\Big|_{\substack{x\in\partial\Omega \\ t\in T}} = \int_0^t \int_\Omega G\big(x, x_0, t, \tau; \mu\big) S(x_0, \tau)\, dx_0\, d\tau$$

CONDITIONAL PROBABILITY

Assume $u_0 \approx Z \quad gaussian \qquad u_0(x,t)\big|_{\substack{x\in\partial\Omega \\ t\in T}} = \int_0^t \int_\Omega G(x,x_0,t,\tau;\mu)S(x_0,\tau)dx_0 d\tau$

$$\Pr\left[\mu(x,t) = \mu / u_0(x_0,t_0) = U_0\right] \qquad ?$$

$$\overline{\mu/u_0} = \int p(\mu/u_0)\mu\, d\mu$$

$$= a_0 u_0 + a_1 u_0^2 + a_2 u_0^3 \ldots\ldots\ldots$$

$$= a_0 Z + a_1 Z^2 + a_2 Z^3 + \ldots\ldots$$

Assume $\mu = \sum b_n H_n(Z) \qquad H_n(Z) \quad Hermite$

Solve $\left\langle \left[Z - \int_0^t \int_\Omega G\left(x,x_0,t,\tau;\sum a_n H_n(Z) \right) S(x_0,\tau)dx_0 d\tau \right]^2 \right\rangle \quad \min$

STOCHASTIC MULTI-SCALE APPROACH

$$u_0(x,t)\Big|_{\substack{x\in\partial\Omega \\ t\in T}} = \int_0^t\int_\Omega G(x,x_0,t,\tau;\mu)S(x_0,\tau)dx_0 d\tau$$

Assume

$$u_0 = \overline{u}_0 + \delta u_0 \qquad \mu = \overline{\mu} + \delta\mu$$

Large scale equation

$$\overline{u}_0(x,t)\Big|_{\substack{x\in\partial\Omega \\ t\in T}} = \int_0^t\int_\Omega G(x,x_0,t,\tau;\overline{\mu})S(x_0,\tau)dx_0 d\tau + \int_0^t\int_\Omega \frac{\partial^2 G(x,x_0,t,\tau;\overline{\mu})}{\partial\mu^2}\overline{\delta\mu^2}\, S(x_0,\tau)dx_0 d\tau$$

Small scale equation

$$\delta u_0(x,t)\Big|_{\substack{x\in\partial\Omega \\ t\in T}} = \int_0^t\int_\Omega \frac{\partial(x,x_0,t,\tau;\overline{\mu})}{\partial\mu}\delta\mu\, S(x_0,\tau)dx_0 d\tau + \int_0^t\int_\Omega \frac{1}{2}\frac{\partial^2 G(x,x_0,t,\tau;\overline{\mu})}{\partial\mu^2}\left(\overline{\mu^2} + 2\overline{\mu}\delta\mu\right)S(x_0,\tau)dx_0 d\tau$$

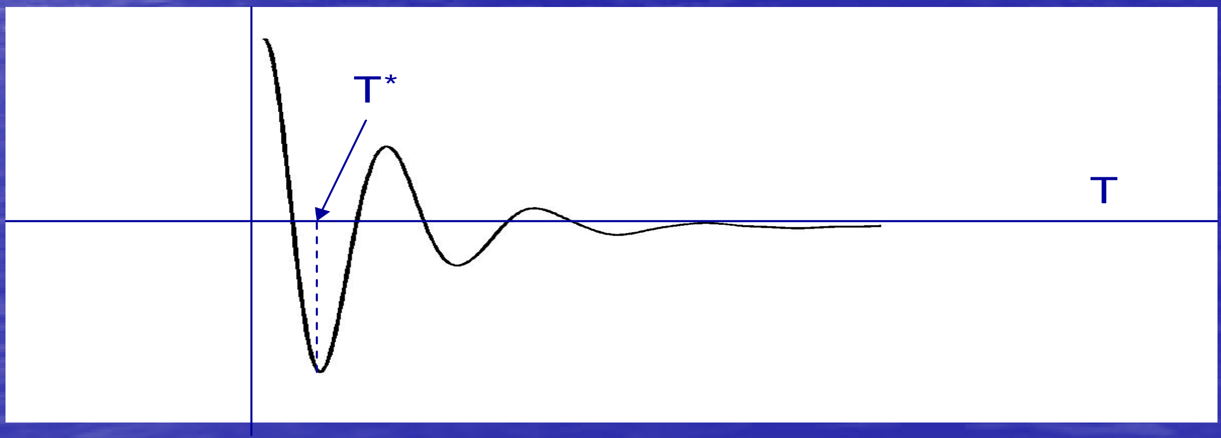
Minimize Large scale equation with constraint $\overline{\delta\mu^2}$ min

PROBABILITY OF EXCEEDANCE

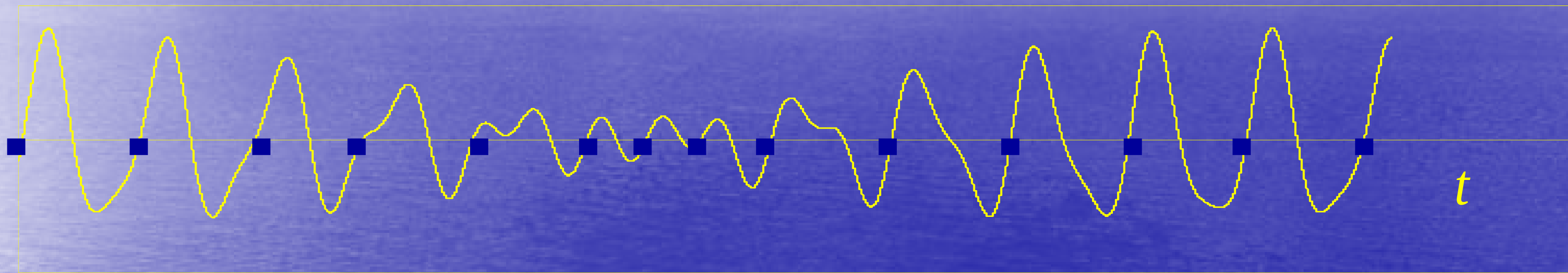
Asymptotic expressions of Boccotti valid for any shape of sprectrum

$$P[H] = \exp\left[-\frac{1}{4(1+\psi^*)}\left(\frac{H}{\sigma}\right)^2\right] \quad per \quad \frac{H}{\sigma} \to \infty$$

ψ^* first minimum of $$\Psi(T) = \frac{\int_0^\infty E(\omega)\cos(\omega T)\,d\omega}{\int_0^\infty E(\omega)\,d\omega}$$

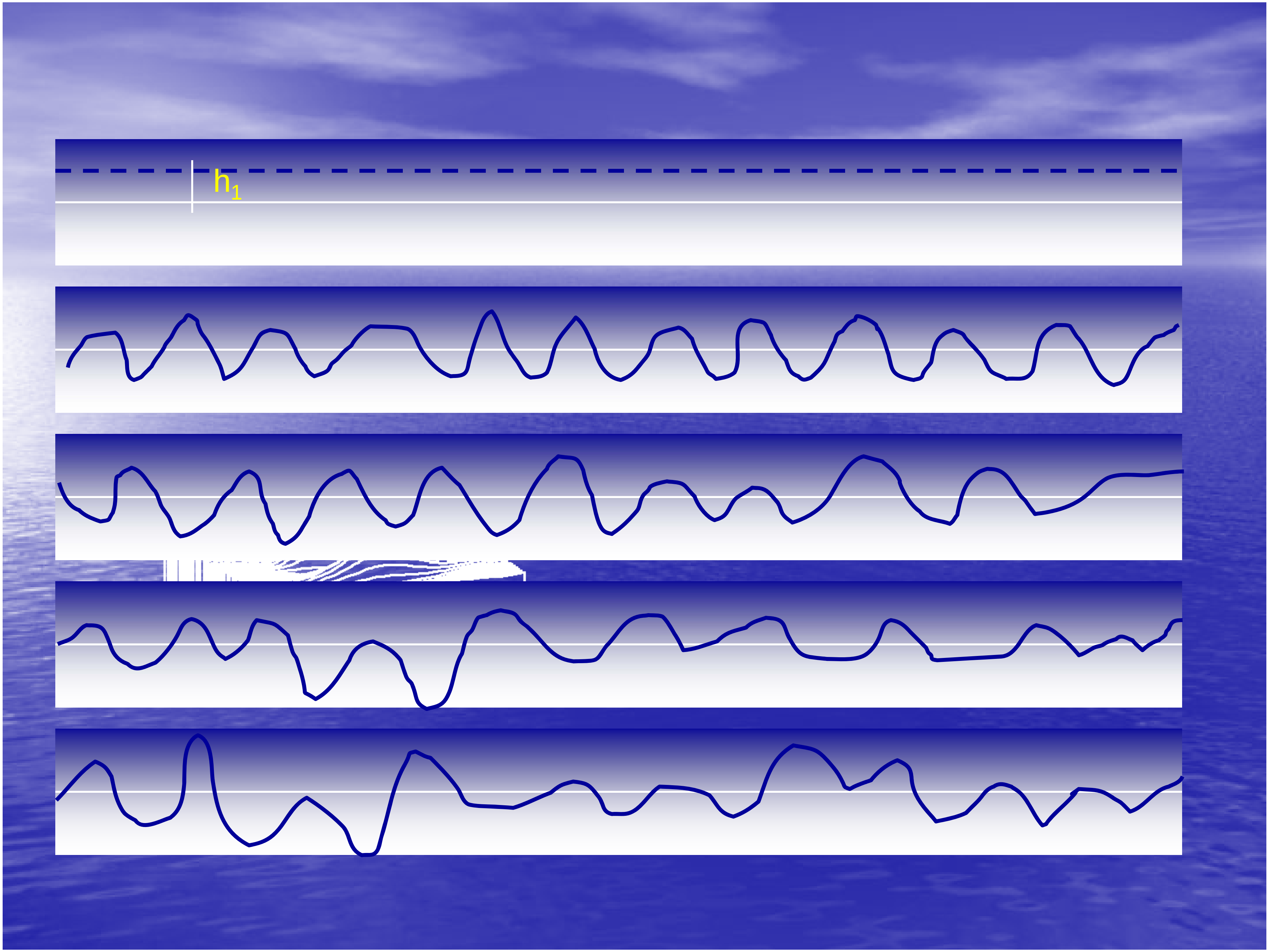


HOW CAN WE INTERPRET THE PROBABILITY OF EXCEEDANCE ?



$$P[H] = \frac{number\ of\ waves\ with\ height\ greater\ than\ H}{total\ number\ of\ waves}$$

$$P[Z] = \frac{number\ of\ waves\ with\ crest\ greater\ than\ Z}{total\ number\ of\ waves}$$


h_1