Felice Arena
Department of Mechanics and Materials,
University 'Mediterranea' of Reggio Calabria,
Loc. Feo di Vito,
89100 Reggio Calabria,
Italy
e-mail: arena@unirc.it

Francesco Fedele
Department of Civil & Environmental Engineering,
University of Vermont,
Votey Building 213,
Burlington, VT 05405
e-mail: ffedele@emba.uvm.edu

Nonlinear Space–Time Evolution of Wave Groups With a High Crest

The theory of quasi-determinism, for the mechanics of linear random wave groups was obtained by Boccotti in the eighties. The first formulation of the theory deals with the largest crest amplitude; the second formulation deals with the largest wave height. In this paper the first formulation of Boccotti's theory, particularized for long-crested waves, is extended to the second-order. The analytical expressions of the nonlinear free surface displacement and velocity potential are obtained. The space–time evolution of the nonlinear wave group, when a very large crest occurs at a fixed time and location, is then shown. Finally the second-order probability of exceedance of the crest amplitude is obtained and validated by Monte Carlo simulation. [DOI: 10.1115/1.1854705]

Keywords: *Surface displacement, Velocity potential, Second-order, Wave crest, Wave group, Probability of exceedance*

Introduction

The theory of quasi-determinism for the mechanics of linear random waves was obtained by Boccotti in the eighties. The first formulation of the theory (Boccotti [1,2]) gives necessary and sufficient conditions for the occurrence of a large wave crest at a certain location in space and in time. Boccotti then extended this first formulation [3–5] to deal with the highest crest-to-trough wave height. He showed that both the highest wave height and the highest wave crest are different occurrences of the space–time evolution of a well defined wave group. Thus the two formulations are congruent to each other [6].

The theory was verified in the nineties, both for waves in an undisturbed field and for waves interacting with structures, by means of some small-scale field experiments (Boccotti [7,8], Boccotti et al., [9,10]).

The quasi-determinism was obtained with a different procedure by Phillips et al. [11,12], which obtained also a field verification off the Atlantic coast of the USA.

A rigorous analysis of the statistical properties of a Gaussian field in the neighborhood of a local maximum was given in the early seventies by Lindgren [13,14].

The first formulation of the theory was also given by Tromans et al. ([15]), which renamed the theory as "New wave."

In this paper we consider unidirectional random waves (long-crested waves) in deep water. The results given here can be easily extended to consider three-dimensional waves on finite depth, but this will not be discussed here. Following the first formulation of the theory of quasi-determinism we derive the analytical expressions of the nonlinear free surface displacements $\overline{\eta}$ and velocity potential $\overline{\phi}$, when a very high crest occurs at a fixed time and location. In particular, the nonlinear surface displacement $\overline{\eta}$ enables us to represents, in space–time domain, the evolution of a nonlinear wave group. The nonlinear dynamics of this group is then investigated.

Finally, the second-order probability of exceeding the largest crest height is obtained. For the case of JONSWAP spectra, it is shown that the new theoretical crest height distribution agrees very well with the numerical distribution from Monte Carlo simulations of second-order random waves.

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The Theory of Quasi-Determinism for a Linear High Crest

The theory of quasi-determinism by Boccotti ([6]), for linear three-dimensional wave groups (either for high wave height or for high wave crest) may be used in place of the periodic waves: It enables us to predict, in space–time domain, the free surface displacement and the velocity potential when a very high wave occurs. The theory may be applied either for waves in an undisturbed field or for waves interacting with structures.

In this paper we shall consider only the first formulation, particularized for long-crested random waves. In detail we have that, if a local wave maximum of given elevation H_C occurs at a time t_o at a fixed point x_o, and if $H_C/\sigma\to\infty$ (σ being the standard deviation of the free surface displacement), with probability approaching 1 the surface displacement at point x_o+X at time t_o+T is asymptotically equal to the deterministic form

$$\overline{\eta_I}(x_o+X,t_o+T)=\frac{\Psi(X,T;x_o)}{\Psi(0,0;x_o)}H_C \quad \text{if} \quad H_C/\sigma\to\infty \tag{1}$$

Let us note that the theory is exact for $H_C/\sigma\to\infty$, that is when the crest height is very large with respect to the mean crest height.

The space–time covariance $\Psi(X,T;x_o)$ is defined as

$$\Psi(X,T;x_o)\equiv\langle\eta(x_o,t)\,\eta(x_o+X,t+T)\rangle \tag{2}$$

where the time average operator $\langle\cdot\rangle$ is defined as

$$\langle f(t)\rangle=\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau f(t)\mathrm{d}t \tag{3}$$

Because the absolute maximum of the autocovariance function $\psi(T)$ is at $T=0$, the local wave maximum of given elevation H_C is the highest maximum of its wave. We have also a symmetric $\overline{\eta_I}$ profile both in time domain (for $X=0$) and in the space domain (for $T=0$).

The velocity potential of the deterministic wave group (1), at point x_o+X, at depth z, is given by

$$\overline{\phi_I}(x_o+X,z,t_o+T)=\frac{\Phi(X,z,T;x_o)}{\Psi(0,0;x_o)}H_C \tag{4}$$

where the space–time covariance $\Phi(X,z,T;x_o)$ is defined as

$$\Phi(X,z,T;x_o)\equiv\langle\eta(x_o,t)\,\phi(x_o+X,z,t+T)\rangle \tag{5}$$

The Linear Deterministic Wave Group in an Undisturbed Field. For long-crested waves in deep water, the free surface displacement [see Eq. (1)] of the wave group at (x_o+X,t_o+T), when an exceptional crest height of given elevation H_C occurs at time t_o at fixed point x_o, may be rewritten as a function of the frequency spectrum $E(\omega)$:

$$\overline{\eta_I}(x_o+X,t_o+T)=\frac{H_C}{\sigma^2}\int_0^\infty E(\omega)\cos(kX-\omega T)\mathrm{d}\omega \qquad (6)$$

where

$$\sigma^2=\int_0^\infty E(\omega)\mathrm{d}\omega \qquad (7)$$

As function of the frequency spectrum, the velocity potential at a fixed point (x_o+X,z), when a very large crest occurs at point x_o, is given by:

$$\overline{\phi_I}(x_o+X,z,t_o+T)=\frac{gH_C}{\sigma^2}\int_0^\infty E(\omega)\omega^{-1}\exp(kz)\sin(kX-\omega T)\mathrm{d}\omega \qquad (8)$$

where the wave number k, in deep water, is equal to ω^2/g. From Eq. (8) one can derive the linear wave kinematics of wave group as well as the first Stokes order pressure fluctuation $\Delta\overline{p_I}$ (which is defined as $\Delta\overline{p_I}=-\rho\partial\overline{\phi_I}/\partial t$).

Nonlinear Space–Time Evolution of a High Wave Crest

According to the theory of wind-generated waves, the free surface displacement to the first-order in a Stokes expansion is a random Gaussian process. The water surface is then modeled as a sum of a very large number of periodic components, with phase angles randomly and uniformly distributed between 0 and 2π. The wave amplitudes follow the Rayleigh distribution.

To the second-order in a Stokes expansion, the free surface displacement and the velocity potential, for long-crested random deep-water waves, are respectively, given by (Sharma and Dean [16], Tayfun [17]):

$$\eta(x,t)=\eta_1+\eta_2=\sum_{n=1}^{N}a_n\cos\psi_n+\frac{1}{4}\sum_{n=1}^{N}\sum_{m=1}^{N}a_na_m[(k_n+k_m)\cos(\psi_n+\psi_m)-|k_n-k_m|\cos(\psi_n-\psi_m)] \qquad (9)$$

$$\phi(x,z,t)=\phi_1+\phi_2=g\sum_{n=1}^{N}a_n\omega_n^{-1}\exp(k_nz)\sin\psi_n-\sum_{n=1}^{N}\sum_{m=n}^{N}a_na_m\omega_m\exp[(k_m-k_n)z]\sin(\psi_m-\psi_n) \qquad (10)$$

where

$$\psi_n=k_nx-\omega_nt+\varepsilon_n \qquad (11)$$

and $\{a_n\}_{n\in\aleph}$, $\{\varepsilon_n\}_{n\in\aleph}$ coefficients to be specified.

In the following we shall derive sufficient and necessary conditions such that $\eta(x,t)$ [see Eq. (9)] attains a local maximum at point $x=x_o$ at time $t=t_o$. By means of the theory of quasi-determinism this maximum is the crest of its own wave (see also Lindgren [13,14]). Thus the deterministic wave group solution to the second-order is derived.

The Deterministic Wave Group to the Second-Order in a Stokes Expansion: The Free Surface Displacement. Let us assume that the free surface displacement has a local maximum h at point $x=x_o$ and that this maximum occurs at time $t=t_o$. In order to obtain the deterministic free surface displacement $\overline{\eta}(X,T)$ at point x_o+X at time instant t_o+T, we shall follow the approach proposed by Fedele and Arena [18]. Here, h is the crest amplitude and σ the standard deviation of the free surface displacement.

A local maximum is attained at time $t=t_o$ (that is $T=0$) and at point $x=x_o$ (that is $X=0$) if the following conditions are satisfied

$$\overline{\eta}(X,T)\ \text{ such that}\begin{cases}\overline{\eta}|_{X=0,T=0}=h\\ (\partial\overline{\eta}/\partial X)|_{X=0,T=0}=0\\ (\partial^2\overline{\eta}/\partial X^2)|_{X=0,T=0}<0\end{cases}. \qquad (12)$$

By applying a perturbation approach we expand the assigned height h as

$$h=h_0+h_1+h_2+\ \ldots \qquad (13)$$

where h_0, h_1, $h_2,\ldots$ are unknown parameters to be determined. We assume that $h_0\propto\sigma$, $h_1\propto\sigma^2,\ldots$, $h_n\propto\sigma^{n+1}$. From the general solution given by Eq. (9), conditions (12) give three equations as follows:

$$\sum_{n=1}^{N}a_n\cos\vartheta_n+\frac{1}{4}\sum_{n=1}^{N}\sum_{m=1}^{N}a_na_m[(k_n+k_m)\cos(\vartheta_n+\vartheta_m)-|k_n-k_m|\cos(\vartheta_n-\vartheta_m)]=h_0+h_1+h_2+\ \ldots \qquad (14)$$

$$-\sum_{n=1}^{N}a_nk_n\sin\vartheta_n+\frac{1}{4}\sum_{n=1}^{N}\sum_{m=1}^{N}a_na_m[-(k_n+k_m)^2\cdot\sin(\vartheta_n+\vartheta_m)+|k_n-k_m|(k_n-k_m)\sin(\vartheta_n-\vartheta_m)]=0 \qquad (15)$$

$$-\sum_{n=1}^{N}a_nk_n^2\cos\vartheta_n+\frac{1}{4}\sum_{n=1}^{N}\sum_{m=1}^{N}a_na_m[-(k_n+k_m)^3\cdot\cos(\vartheta_n+\vartheta_m)+|k_n-k_m|(k_n-k_m)^2\cos(\vartheta_n-\vartheta_m)]<0 \qquad (16)$$

where $\vartheta_n=k_nx_0-\omega_nt_0+\varepsilon_n$. Because we assume $a_n\propto\sigma$, a hierarchy of perturbation equations to the first and to the second-order in a Stokes expansion may be obtained. All the terms in the h expansion higher than the second-order vanish.

i) Perturbation Equations to $O(\sigma)$. To the first-order, Eqs. (14)–(16) give, respectively,

$$O(\sigma)\begin{cases}\sum_{n=1}^{N}a_n\cos\vartheta_n=h_0\\ \sum_{n=1}^{N}a_nk_n\sin\vartheta_n=0\\ \sum_{n=1}^{N}a_nk_n^2\cos\vartheta_n>0\end{cases} \qquad (17)$$

Note that the second and third Equations in (17) are satisfied whatever are the values of the coefficients $\{a_n\}_{n\in\aleph}$ if one imposes that all the harmonic components are in phase, i.e.,

$$\vartheta_n=0\,\forall n. \qquad (18)$$

The first equation in (17) then gives

$$h_0=\sum_n a_n. \qquad (19)$$

Other solutions could exist with some coefficients $\vartheta_n\neq 0$. In the following we shall prove that condition (18) is necessary and sufficient so that an absolute maximum is attained at $x=x_o$ and $t=t_o$. Moreover this is also a local maximum.

From the quasi-determinism theory we know that if a very large crest height $H_C=h_0$ occurs at a fixed point $x=x_o$ at time instant $t=t_o$, the free surface displacement [Eq. (6)] in discrete form is given by

$$\overline{\eta_I}(X,T)=\sum_{n=1}^{N}\tilde{a}_n\cos\tilde{\psi}_n \qquad (20)$$

where

$$\tilde{a}_n = \frac{h_0}{\sigma^2} E(\omega_n)\mathrm{d}\omega_n \tag{21}$$

and

$$\tilde{\psi}_n = k_n X - \omega_n T. \tag{22}$$

Note that the high wave group defined by Eq. (20) at $(X=0, T=0)$ attains a local maximum since the first derivative in space vanishes and the second derivative is negative at that location, i.e.,

$$(\partial \overline{\eta_I}/\partial X)_{X=0,T=0} = -\sum_{n=1}^{N} \tilde{a}_n k_n \sin \tilde{\psi}_n = 0 \tag{23}$$

$$(\partial^2 \overline{\eta_I}/\partial X^2)_{X=0,T=0} = -\sum_{n=1}^{N} \tilde{a}_n k_n^2 \cos \tilde{\psi}_n < 0 \tag{24}$$

Moreover, since at $X=0$, $T=0$ we have $\tilde{\psi}_n = 0 \forall n$, this yields the wave amplitude at that location, that is

$$\overline{\eta_I}(X=0, T=0) = \sum_{n=1}^{N} \tilde{a}_n = h_0. \tag{25}$$

Note that in the linear case h_0 is the highest amplitude that can be attained for the assigned spectrum $E(\omega)$ or equivalently for the assigned harmonic components $\tilde{a}_n$. Thus at $X=0$, $T=0$ the local maximum is also an absolute maximum. Equations (23)–(25) are identical to the system of Eqs. (17) if

$$\vartheta_n = \tilde{\psi}_n(X=0, T=0) = 0 \tag{26}$$

and

$$a_n = \tilde{a}_n \tag{27}$$

where $\tilde{a}_n$ is given by Eq. (21). Thus, the linear wave group $\overline{\eta_I}$ is identical to the first-order part of the nonlinear group $\overline{\eta}$ if the conditions (26) and (27) hold. Then the condition $\vartheta_n = 0 \forall n$ is necessary and sufficient for the existence of an absolute maximum at $X=0$, $T=0$ which is also a local maximum. Equation (27) gives the amplitude of the harmonic coefficients a_n as a function of the assigned frequency spectrum, that is

$$a_n = \frac{h_0}{\sigma^2} E(\omega_n)\mathrm{d}\omega_n. \tag{28}$$

ii) The Second-Order Problem. To the second-order, Eqs. (14)–(16) give

$$O(\sigma^2)\begin{cases} \frac{1}{4}\sum_{n,m} a_n a_m[(k_n+k_m)\cos(\vartheta_n+\vartheta_m) - |k_n-k_m|\cos(\vartheta_n-\vartheta_m)] = h_1 \\ \frac{1}{4}\sum_{n,m} a_n a_m[-(k_n+k_m)^2\sin(\vartheta_n+\vartheta_m) + |k_n-k_m|(k_n-k_m)\sin(\vartheta_n-\vartheta_m)] = 0 \\ \frac{1}{4}\sum_{n,m} a_n a_m[-(k_n+k_m)^3\cos(\vartheta_n+\vartheta_m) + |k_n-k_m|\cdot(k_n-k_m)^2\cos(\vartheta_n-\vartheta_m)] < \sum_{n=1}^{N} a_n k_n^2 \cos\vartheta_n \end{cases} \tag{29}$$

From the first-order problem [Eqs. (17)], it has been shown that $\vartheta_i = 0 \forall i$, which implies that the last two conditions in Eq. (29) are satisfied, while the first condition becomes the following form

$$h_1 = \frac{1}{4}\sum_{n,m} a_n a_m[(k_n+k_m) - |k_n-k_m|] \tag{30}$$

By considering Eq. (21), which defines a_n, we obtain, in continuous form

$$h_1 = \frac{H_C^2}{4\sigma^4}\int_0^\infty\int_0^\infty E(\omega_1)E(\omega_2)[(k_1+k_2) - |k_1-k_2|]\mathrm{d}\omega_1\mathrm{d}\omega_2 \tag{31}$$

Finally, we have that, if a very large crest height occurs, the second-order height may be written as:

$$h = H_C + \frac{H_C^2}{4\sigma^4}\int_0^\infty\int_0^\infty E(\omega_1)E(\omega_2)[(k_1+k_2) - |k_1-k_2|]\mathrm{d}\omega_1\mathrm{d}\omega_2 + o(\sigma^2) \tag{32}$$

where symbol $o(\sigma^n)$ indicates terms which are of order greater than n.

In general, the second-order free surface displacement, when a very high crest occurs at time instant t_o at point x_o, is given by:

$$\overline{\eta}(X,T) = \overline{\eta_I} + \overline{\eta_{II}} = \frac{H_C}{\sigma^2}\int_0^\infty E(\omega)\cos\tilde{\psi}\mathrm{d}\omega + \frac{H_C^2}{4\sigma^4}\int_0^\infty\int_0^\infty E(\omega_1)E(\omega_2)[(k_1+k_2)\cos(\tilde{\psi}_1+\tilde{\psi}_2) - |k_1-k_2|\cos(\tilde{\psi}_1-\tilde{\psi}_2)]\mathrm{d}\omega_1\mathrm{d}\omega_2 \tag{33}$$

The Deterministic Wave Group to the Second-Order in a Stokes Expansion: The Velocity Potential. The velocity potential of the linear wave group (8) may be written in discrete form as

$$\overline{\phi_I}(X,z,T) = g\sum_{n=1}^{N} \tilde{a}_n \omega_n^{-1}\exp(k_n z)\sin\tilde{\psi}_n \tag{34}$$

By considering the general second-order solution for the velocity potential, given by Eq. (10), we obtain the second-order velocity potential of the wave group in continuous form as

$$\overline{\phi}(X,z,T) = \overline{\phi_I} + \overline{\phi_{II}} = \frac{gH_C}{\sigma^2}\int_0^\infty E(\omega)\omega^{-1}\exp(kz)\sin\tilde{\psi}\mathrm{d}\omega + \frac{H_C^2}{\sigma^4}\cdot\int_0^\infty\int_{\omega_1}^\infty E(\omega_1)E(\omega_2)\omega_2\exp[(k_2-k_1)z]\times\sin(\tilde{\psi}_1-\tilde{\psi}_2)\mathrm{d}\omega_2\mathrm{d}\omega_1 \tag{35}$$

Calculation of the Second-Order $\overline{\eta}$ and $\overline{\phi}$

Let us consider the JONSWAP spectrum (Hasselmann et al. [19]), which is defined as follows

$$E(w\omega_p)=\alpha g^2\omega_p^{-5}E_a(w), \tag{36}$$

where α is the Phillips parameter, $\omega_p=2\pi/T_p$ the peak frequency, and

$$E_a(w)=w^{-5}\exp[-1.25w^{-4}]\exp\left\{\ln\chi_1\exp\left[-\frac{(w-1)^2}{2\chi_2^2}\right]\right\} \tag{37}$$

is the nondimensional spectrum (being $w_j=\omega_j/\omega_p$).Typical values of the spectrum parameters are $\chi_1=3.3$ and $\chi_2=0.08$ (mean JONSWAP).

Furthermore, by defining $k_j=k_{w_j}2\pi/L_{p_o}$, where L_{p_o} $[\equiv gT_p^2/(2\pi)]$ is the wavelength in deep water, the nondimensional wave number is $k_{w_j}=w_j^2$ and, therefore, $\psi_j=k_{w_j}2\pi X/L_{p_o}-2\pi w_jT/T_p$.

It follows that

$$\begin{aligned}\overline{\eta}(X,T)=&\frac{H_C}{\sigma_w^2}\int_0^\infty E_a(w)\cos\tilde{\psi}\mathrm{d}w\\&+\frac{H_C^2}{4\sigma_w^4}\frac{\omega_p^2}{g}\int_0^\infty\int_0^\infty E_a(w_1)E_a(w_2)[(w_1^2+w_2^2)\\&\times\cos(\tilde{\psi}_1+\tilde{\psi}_2)-|w_1^2-w_2^2|\cos(\tilde{\psi}_1-\tilde{\psi}_2)]\mathrm{d}w_1\mathrm{d}w_2\end{aligned} \tag{38}$$

$$\begin{aligned}\overline{\phi}(X,z,T)=&\frac{gH_C}{\sigma_w^2}\omega_p^{-1}\int_0^\infty E_a(w)w^{-1}\exp(2\pi k_wz/L_{p0})\cdot\sin\tilde{\psi}\mathrm{d}w\\&+\frac{H_C^2}{\sigma_w^4}\omega_p\int_0^\infty\int_{\omega_1}^\infty E_a(w_1)E_a(w_2)w_2\\&\cdot\exp[(k_{w2}-k_{w1})2\pi z/L_{p0}]\sin(\tilde{\psi}_1-\tilde{\psi}_2)\mathrm{d}w_2\mathrm{d}w_1\end{aligned} \tag{39}$$

where

$$\sigma_w^2=\int_0^\infty E_a(w)\mathrm{d}w=\sigma^2/(\alpha g^2\omega_p^{-4}) \tag{40}$$

Let us note that, from the second-order velocity potential (39), we may easily derive the wave pressure and the wave kinematics, exact to the second-order in a Stokes expansion (see also Jensen [20]). As an example, the second-order pressure fluctuation is $\Delta\overline{p}=\Delta\overline{p}_I+\Delta\overline{p}_{II}$, where the second-order component $\Delta\overline{p}_{II}$ is easily derived by the formula

$$\Delta\overline{p}_{II}=-\rho\frac{\partial\overline{\phi}_{II}}{\partial T}-\frac{1}{2}\rho\left[\left(\frac{\partial\overline{\phi}_I}{\partial X}\right)^2+\left(\frac{\partial\overline{\phi}_I}{\partial z}\right)^2\right] \tag{41}$$

(see Applications).

Applications

The Wave Crest Evolution in Space Domain. Figure 1 shows the space–time evolution of the second-order free surface displacement, when a very large crest height occurs at point x_o at time instant t_o. In particular, the deep-water second-order free surface displacement, computed by means of Eq. (38), is shown *in the space domain*, at some fixed time instant. These graphs of the water surface emphasize the existence of a well defined wave group, which moves along the x-axis, crossing the point x_o (where $X=0$).

The space–time evolution of this nonlinear wave group is similar to the linear dynamics of the group given by Boccotti ([6]). Since the propagation speed for individual waves is greater than the wave group celerity, each wave "runs along the envelope from the tail where it is born to the head where it dies" (Boccotti [6]). In fact from Fig. 1, the wave group shows firstly a development stage, during which the height of the largest crest (at that fixed instant) increases; therefore at time t_o we have the apex of the

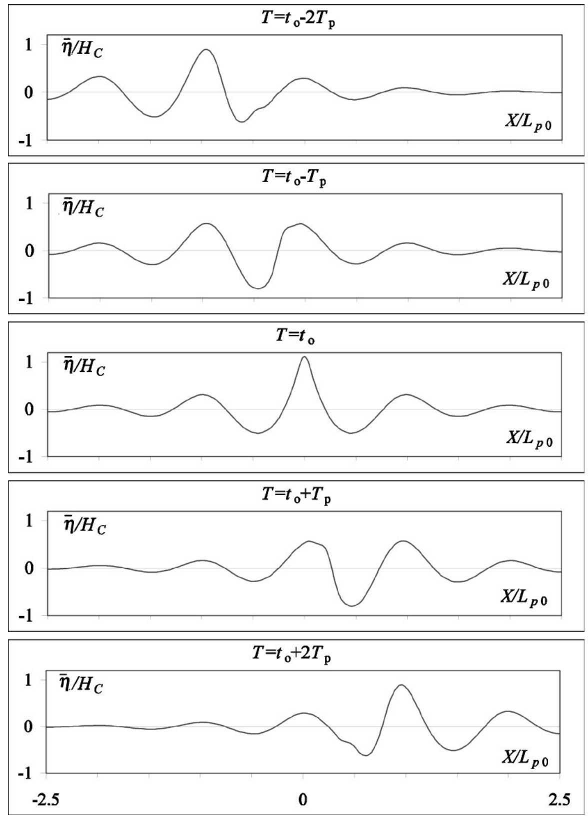

Fig. 1 The second-order space–time evolution of a wave group in which a very large crest occurs at $(X=0,T=t_o)$

group development: at this time the wave crest at point x_o reaches its maximum; finally the wave group has a decay stage.

Second-order effects are evident, *in the space domain*, at time t_o, because the largest crest and trough amplitudes are equal, respectively, to 1.11 and 0.94 times the linear predictions. Moreover the ratio between largest crest and trough amplitudes is equal to 1.87 for the linear group and to 2.22 for the second-order group.

The Second-Order Wave Crest Evolution in Time Domain Figures 2–5 show a wave group *in the time domain*. Figure 2 shows the wave group at point x_o, where the largest crest occurs. Dotted line gives the linear prediction, which is computed by means of Eq. (6): we have the well known symmetric profile ("New wave"), obtained also by Tromans et al. ([15]). In this case the linear largest trough amplitude is equal to ψ^* times the largest crest amplitude, being ψ^* the narrow bandedness param-

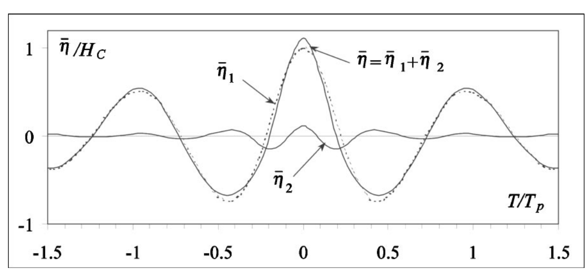

Fig. 2 The second-order time evolution of a wave group in which a very large crest occurs. The solid lines give the second-order prediction [Eq. (38)]. The dotted line gives linear prediction [Eq. (6)], which is the "New wave" symmetric wave profile.

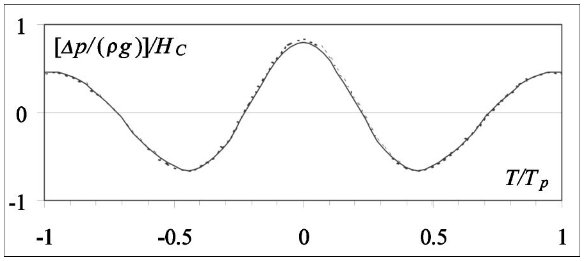

Fig. 3 The time domain second-order wave pressure Δp at point $(x_o, z/L_{p0}=-0.05)$, when a very large crest height H_C of free surface displacement occurs at $(x_o, T=0)$ (see Fig. 2). The solid lines give the second-order perdiction. The dotted line gives linear prediction.

eter defined by Boccotti ([3,4]). The parameter ψ^* is defined as the absolute value of the quotient between the absolute minimum and the absolute maximum of the autocovariance function. It is equal to 0.73 for the mean JONSWAP spectrum and to 0.67 for a Pierson–Moskowitz frequency spectrum. The ratio between crest

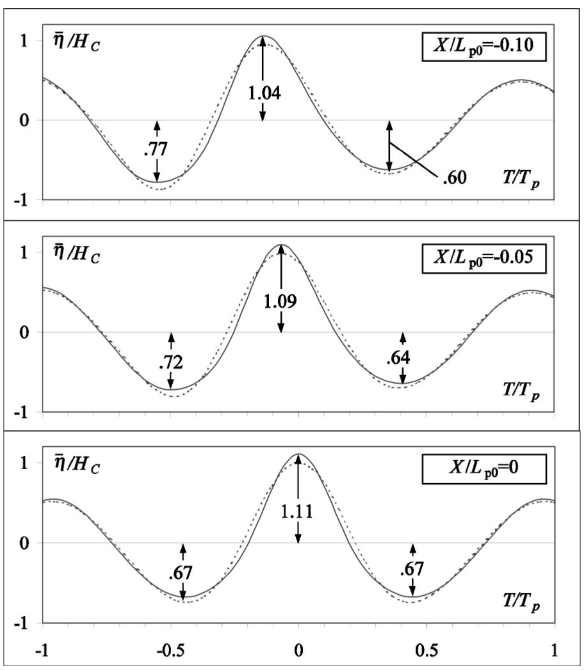

Fig. 4 The second-order time evolution of a wave group at fixed points X/L_{p0}, when a very large crest occurs at $(X=0, T=0)$. The dotted lines show linear predictions, obtained from Boccotti's quasi-determinism theory (first formulation).

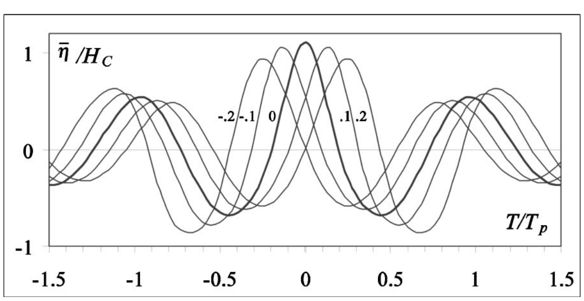

Fig. 5 The second-order time evolution of a wave group at fixed points X/L_{p0} $(-0.20,-0.10,0,0.10,0.20)$ when a very large crest occurs at $(X=0, T=0)$

and trough amplitudes is then 1.37 and 1.49, respectively. To the second-order, *in the time domain*, the ratio between crest and trough amplitudes is equal to 1.67 for the mean JONSWAP spectrum and to 1.85 for the Pierson–Moskowitz spectrum.

The second-order wave profile $\overline{\eta}(x_o, t)$ (being $\overline{\eta}=\overline{\eta_1}+\overline{\eta_2}$) is furthermore symmetric, because the component $\overline{\eta_2}$ is symmetric. The highest wave crest is in phase with the linear highest crest of $\overline{\eta_I}$.

Figure 3 shows the second-order wave pressure $\Delta\overline{p}$ at point $(x_o, z=-0.05L_{p0})$, when a very large surface crest occurs at point x_o. As one can see from Fig. 3, second-order effects slightly reduce the amplitude of the highest crest of the linear component $\Delta\overline{p_I}$.

Asymmetries in the Time Domain Wave Profile. By applying the first formulation of the quasi-determinism theory we may obtain the free surface displacement in time domain at any fixed point x_o+X, if a very large crest occurs at point x_o. In Fig. 2 we have seen the linear wave profile at point x_o in time domain, which is a symmetric profile (New wave).

As discussed above the second-order wave profile is also symmetric, either in space domain for $T=0$ or in time domain for $X=0$. Here we analyze the wave profile $\overline{\eta}(T)$ at some points close to x_o. Figure 4 shows the wave profile $\overline{\eta}(T)$ at the locations $X/L_{po}=(-0.10,-0.05,0)$. Note that, for $X\neq 0$ the profile is not symmetric: The trough depths before (H_{T1}) and after (H_{T2}) the highest crest are different, with $H_{T1}>H_{T2}$ (let us note that at points having $X/L_{p0}>0$ we have $H_{T1}<H_{T2}$, as one can see from Fig. 5).

Thus, the asymmetry in the wave profile may be explained by analyzing the space–time evolution of the wave groups: For example, if we have a time record with a very high crest, with $H_{T1}>H_{T2}$ $(H_{T1}<H_{T2})$, we have then probably recorded the highest crest at a point $X<0$ $(X>0)$, before (after) point x_o where the wave group reaches the apex of its development.

Finally, it is easy to verify that in deep water these asymmetries are slightly modified by second-order nonlinearities: They may be explained by the linear quasi-determinism theory, from Eq. (6).

The Asymptotic Form of the Second-Order Probability of Exceedance of the Crest Height

Analytical models for the nonlinear wave crest distribution have been proposed by many authors (see Tayfun [17], Tung and Huang [21], Kriebel and Dawson [22], Forristall [23], Prevosto et al. [24], Arena and Fedele [25], Al-Humond et al. [26], Tayfun and Al-Humoud [27]).

Fedele and Arena [18,28], based on the second-order crest amplitude given by Eq. (32) derived an asymptotic distribution law which is valid for finite-band spectra. They showed that the nonlinear crest may be rewritten as

$$h=H_C+\varphi\frac{H_C^2}{\sigma} \tag{42}$$

where

$$\varphi=\frac{\varepsilon_p}{4\sigma_w^4}\int_0^\infty\int_0^\infty E_a(w_1)E_a(w_2)[(w_1^2+w_2^2)-|w_1^2-w_2^2|]\mathrm{d}w_1\mathrm{d}w_2 \tag{43}$$

being $\varepsilon_p=k_p\sigma$ the wave steepness $(k_p\equiv 2\pi/L_{p0})$. The variance of the second-order process is easily derived from Eq. (9) and has expression as

$$\sigma_\eta^2=\frac{\sigma^2}{\beta^2} \tag{44}$$

where

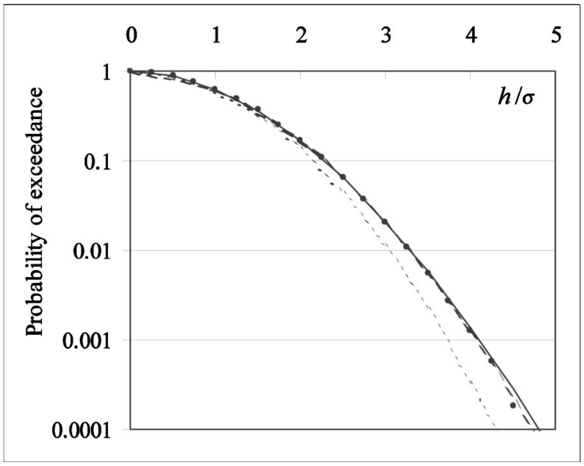

Fig. 6 The second-order probability of exceeding the crest height, obtained both with presented model (continuous line) and with Forristall model (broken line). The dotted line gives the Rayleigh distribution (exact to the first-order). Data is obtained from numerical simulations.

$$\beta = \left[1 + \frac{\varepsilon_p^2}{2\sigma_w^4} \int_0^\infty \int_0^\infty E_a(w_1) E_a(w_2)(w_1^4 + w_2^4) \mathrm{d}w_1 \mathrm{d}w_2 \right]^{-1/2} \quad (45)$$

The nondimensional crest height $\xi_{\text{high}} = h/\sigma_\eta$ can then be expressed as

$$\xi_{\text{high}} = \beta u + \varphi \beta u^2 \quad (46)$$

where the random variable $u = H_C/\sigma$ has Rayleigh distribution. Consequently after some algebra we obtain the probability of exceeding the absolute maximum (crest) as

$$P(\xi_{\text{high}} > \xi) = \exp\left[-\frac{1}{8\varphi^2} \left(1 - \sqrt{1 + \frac{4|\varphi|\xi}{\beta}} \right)^2 \right] \quad (47)$$

The probability $P(\xi_{\text{high}} > \xi)$, which is valid for $\xi \to \infty$, depends upon the two parameters φ and β.

The Distribution of the Crest of the Highest Waves: Analytical Prediction and Comparison with Data. The analytical prediction of the crest height distribution is then compared with the data of numerical simulations. In detail a second-order simulation of random waves with a mean JONSWAP spectrum has been carried out, with a generation of nearly 50000 waves.

Figure 6 shows a good agreement between the crest height distribution obtained from data and the theoretical prediction given by Eq. (47). Let us note that for the mean JONSWAP spectrum, in deep water, we have $\varphi = 0.028$ and $\beta = 0.996$. Finally, Fig. 6 shows that theoretical prediction are in good agreement with the crest height distribution obtained with the Forristall model.

Conclusions

The first formulation of the theory of quasi determinism by Boccotti, has been extended to take into account the second-order effects. The nonlinear free surface displacements $\overline{\eta}$ and velocity potential $\overline{\phi}$, when a very high crest occurs at a fixed time and location, have been derived.

Second-order effects have been analyzed, both in space domain (at some fixed time instant) and in time domain (by analyzing the wave profile and its asymmetries when a large crest occurs).

Finally, the second-order probability of exceeding the largest crest height has been obtained. For the case of JONSWAP spectra, it has been shown that the new theoretical crest height distribution agrees very well with the numerical distribution from Monte Carlo simulations of the second-order random waves.

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