The Occurrence of Extreme Crests and the Nonlinear wave-wave Interaction in Deep-water Random Seas

Francesco Fedele
Department of Civil and Environmental Engineering
University of Vermont, Burlington, Vermont USA

ABSTRACT

In this paper, sufficient conditions for the occurrence of an extreme crest in weakly nonlinear water waves are given. The starting point is the Zakharov equation which governs the dynamics of the spectral components of the wave envelope of the surface displacement $\eta(\mathsf{x}, t)$. It is proven that the optimal spectral components giving an extreme crest at $(\mathsf{x} = \mathsf{0}, t = 0)$ are solutions of a well defined constrained optimization problem. A new analytical expression for the probability of exceedance of the wave crest is then derived by means of the theory of quasi-determinism of Boccotti. The analytical results agree well with measurements data at the Draupner field and can be used for the prediction of freak wave events.

Key words: Extreme crest, Zakharov equation, wave-wave interaction, quasi-determinism, energy transfer, probability of exceedance, freak wave.

INTRODUCTION

Single waves that are extremely unlikely as judged by the Raleigh distribution are called freak waves. The freak event occurred on January 1^{st} 1995 under the Draupner platform in the North Sea (Wist et al., 2002) provides evidence that such waves can occur in the open ocean. During this freak event, an extreme crest with an amplitude of 18.5 m occurred. The maximal wave height of 25.6 m was much more than twice the significant wave height of about 10.8 m.

Two linear mechanisms which can cause such a concentration or focusing of wave energy in a small area of the ocean have been proposed: time-space focusing and current focusing. In particular, the first mechanism can be explained by means of the theory of quasi-determinism (Boccotti, 1981,1982,1989,1995,1997,2000). Boccotti showed that, if in a Gaussian sea a very high wave height occurs at some point in space and time, with high probability a well defined quasi-deterministic wave group generates the high crest. In particular, the initial configuration of the wave group is such that in earlier stage of evolution of the group itself, the shortest waves are in front of the long waves. As time evolves, the long waves propagate faster and will catch up on the shorter waves, producing a focusing of energy at some point in space and time. The second linear mechanism, investigated in White & Fornberg (1998), requires almost unidirectional waves entering a zone of variable currents. However, ocean waves have a natural directional spreading, therefore the focusing effect is attenuated. A third mechanism which can be a cause of freak waves is related to the four-wave interaction in weakly nonlinear water waves (Janseen, 2003; Longuet-Higgins, 1962,1976; Phillips, 1961; Benney, 1962; Komen et al., 1996). Weakly nonlinear energy transfer among non-resonant and resonant quartets is governed by the deterministic Zakharov integraldifferential equation. A corrected version of this equation has been derived in Krasitskii (1990,1994) using a Hamiltonian approach and canonical transformations. Under the assumption of narrow-band spectrum, the Zakharov equation reduces down to the nonlinear Schrödinger (NLS) equation (Janseen, 2003). An enhanced NLS equation valid for broader spectral bandwidth and larger steepness has been proposed by Trulsen et al. (2000,2003). Based on this model, the effects of directional spreading on the occurrence of freak waves have been investigated in Onorato et al. (2002). As the directional spreading increases, the appearance of extreme events tends to reduce.

In this paper, sufficient conditions for the occurrence of an extreme crest in weakly nonlinear water waves are given. The starting point is the Zakharov equation which governs the dynamics of the spectral components $B_n(t)$ of the wave envelope of the surface displacement $\eta(\mathsf{x}, t)$. The Zakharov equation admits motion integrals: in particular, its Hamiltonian $\mathcal{H}$ (wave energy), the wave action $\mathcal{A}$ and wave momentum $\mathcal{M}$. It is shown that the spectral components giving an extreme crest at $(\mathsf{x} = \mathsf{0}, t = 0)$ satisfy a well defined constrained optimization problem. As an application, the case of unidirectional deep water waves is considered. In this case the formation of an extreme wave is due to non-resonant energy transfer as the Benjamin-Feir instability. An asymptotic solution for the probability of exceedance of the crest height is then derived. Finally, the analytical predictions are compared to the measurements data at the Draupner field (Wist et al., 2002) .

THE ZAKHAROV EQUATION

Let us consider water waves over a finite depth d. The Zakharov integrodifferential equation for weakly nonlinear water waves to the third order in amplitude, is

$$\frac{\partial B(\mathsf{k},t)}{\partial t} + i\omega(\mathsf{k})B(\mathsf{k},t) = \tag{1}$$

$$= \quad -i\int\int\int T(\mathsf{k},\mathsf{k}_1,\mathsf{k}_2,\mathsf{k}_3)\delta(\mathsf{k}+\mathsf{k}_1-\mathsf{k}_2-\mathsf{k}_3)\cdot$$

$$\cdot B^*(\mathsf{k}_1,t)B(\mathsf{k}_2,t)B(\mathsf{k}_3,t)d\mathsf{k}_1 d\mathsf{k}_2 d\mathsf{k}_3$$

Here the kernel T is a real function of $\mathsf{k},\mathsf{k}_1,\mathsf{k}_2,\mathsf{k}_3$ and is obtained by symmetrization as described in Krasitskii (1994). The coefficient $B(\mathsf{k},t)$ can be interpreted as the spectral component of the wave envelope. Moreover the first order free-surface $\eta(\mathsf{x},t)$ is related to $B(\mathsf{k},t)$ through

$$\eta(\mathsf{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sqrt{\frac{\omega(\mathsf{k})}{2g}}B(\mathsf{k},t)e^{i\mathsf{k}\cdot\mathsf{x}}d\mathsf{k}+c.c. \tag{2}$$

where g^* denotes the complex conjugate of g, k is the wave vector, $\mathsf{x} = (x,y)$ is the horizontal spatial vector and ω is the linearized wave frequency related to k through the linear dispersion relation $\omega^2(\mathsf{k})/g{=}|\mathsf{k}|\tanh(|\mathsf{k}|\,d)$ with g as the acceleration due to gravity. The Zakharov equation is invariant under the transformation $B \to B^*, t \to -t$ and admits as a motion integral, besides its Hamiltonian $\mathcal{H}$ (wave energy)

$$\mathcal{H} \quad = \quad \int \omega(\mathsf{k})B(\mathsf{k},t)B^*(\mathsf{k},t)d\mathsf{k}+ \tag{3}$$

$$+\frac{1}{2}\int\int\int\int T(\mathsf{k},\mathsf{k}_1,\mathsf{k}_2,\delta(\mathsf{k}+\mathsf{k}_1-\mathsf{k}_2-\mathsf{k}_3)\cdot$$

$$\cdot B^*(\mathsf{k},t)B^*(\mathsf{k}_1,t)B(\mathsf{k}_2,t)B(\mathsf{k}_3,t)d\mathsf{k}d\mathsf{k}_1 d\mathsf{k}_2 d\mathsf{k}_3,$$

the wave action $\mathcal{A}$ and wave the momentum $\mathcal{M}$ given by

$$\mathcal{A} = \int B(\mathsf{k},t)B^*(\mathsf{k},t)d\mathsf{k} \qquad \mathcal{M} = \int \mathsf{k}B(\mathsf{k},t)B^*(\mathsf{k},t)d\mathsf{k}. \tag{4}$$

If one considers $B(\mathsf{k},t)$ as the superimposition of discrete modes

$$B(\mathsf{k},t) = \sum\nolimits_n B_n(t)\delta(\mathsf{k}-\mathsf{k}_n) \tag{5}$$

then substituting Eq. (5) into Eq. (1), yields the set of time-varying differential equations

$$\frac{dB_n}{dt} + i\omega_n B_n \tag{6}$$

$$= \quad -i\sum_{p,q,r} T_{npqr}\delta_{n+p-q-r}B_p^* B_q B_r.$$

Here, $\omega_n = \omega(\mathsf{k}_n)$ and $T_{npqr} = T(\mathsf{k}_n,\mathsf{k}_p,\mathsf{k}_q,\mathsf{k}_r)$. The generalized Kronecker delta $\delta_{n+p-q-r}$ denotes that summation is taken over those subscripts satisfying

$$\mathsf{k}_n + \mathsf{k}_p = \mathsf{k}_q + \mathsf{k}_r. \tag{7}$$

Eq. (6) admits the discrete Hamiltonian

$$\mathcal{H} \quad = \quad \sum\nolimits_n \omega_n B_n(t)B_n^*(t)+ \tag{8}$$

$$+\frac{1}{2}\sum\nolimits_{n,p,q,r} T_{npqr}\delta_{n+p-q-r}\cdot$$

$$B_n^*(t)B_p^*(t)B_q(t)B_r(t)$$

and the discrete version of the motion integrals (4) is

$$\mathcal{A} = \sum\nolimits_n B_n(t)B_n^*(t) \qquad \mathcal{M} = \sum\nolimits_n \mathsf{k}_n B_n(t)B_n^*(t). \tag{9}$$

In addition to the constraint (7), if the modes forms a resonant quartet, i.e. $\omega_n{+}\omega_p = \omega_q{+}\omega_r$, the Hamiltonian reduces to $\mathcal{H} = \sum \omega_n |B_n|^2$. In deep water, resonant quartets can only occur for three dimensional waves.

SUFFICIENT CONDITIONS FOR THE OCCURRENCE OF AN EXTREME CREST

If the nonlinear effects are weak enough to be neglected, the free-surface is expressed as follows

$$\eta_L(\mathsf{x},t) = \frac{1}{2\pi}\sum_{n=1}^{N}\sqrt{\frac{\omega_n}{2g}}\tilde{B}_n\exp\left[i(\mathsf{k}_n\cdot\mathsf{x}+\tilde{\varphi}_n)\right] + c.c. \tag{10}$$

Here $\tilde{\varphi}_n$ are arbitrary phase angles and $\tilde{B}_n$ are positive coefficients defining the wave spectrum

$$E(\mathsf{k},t)d\mathsf{k} = \frac{\omega_n}{\pi^2 g}\tilde{B}_n^2\;\delta(\mathsf{k}-\mathsf{k}_n) \tag{11}$$

where N is the number of the spectral components. If the nonlinear effects are not negligible, the free-surface $\eta(\mathsf{x},t)$ is given by

$$\eta(\mathsf{x},t) = \frac{1}{2\pi}\sum_{n=1}^{N}\sqrt{\frac{\omega_n}{2g}}\,|B_n(t)|\exp\left[i\left(\mathsf{k}_n\cdot\mathsf{x}+\varphi_n(t)\right)\right]+c.c. \tag{12}$$

where the spectral component $B_n(t)$ has been expressed as

$$B_n(t) = |B_n(t)|\exp[i\varphi_n(t)] \qquad n = 1,...N$$

with $\varphi_n(t)$ arbitrary time-varying phase angles. Eq. (12) is the superimposition of N harmonic components nonlinearly interacting among each other, according to the evolution equation (6). As time varies, a nonlinear energy transfer among the N harmonic components occurs and the wave energy, action and momentum are conserved [see Eqs. (8) and (9)]. At some initial time $t = -t_0$ the N harmonic components $B_n(t)$ are set to be equal to the corresponding linear components, that is

$$B_n(t=-t_0) = \tilde{B}_n\exp(i\tilde{\varphi}_n) \qquad n = 1,...N.$$

For linear waves, if all the harmonic components are in phase, i.e. $\tilde{\varphi}_n = 0$ $n = 1,...N$, then the surface displacement $\eta_L(\mathsf{x},t)$

has a stationary point at $(\mathsf{x}=\mathsf{0}, t=0)$ where $\eta_L(\mathsf{x},t)$ attains its absolute maximum (Fedele & Arena, 2003)

$$(H_{\mathsf{max}})_{linear} = \frac{1}{\pi}\sum_{n=1}^{N}\sqrt{\frac{\omega_n}{2g}}\tilde{B}_n.$$

The highest wave crest in the linear regime can only occur if all the harmonic components are in phase so that a linear focusing of energy is attained at a certain point in the space. In the nonlinear regime, if one imposes that at $(\mathsf{x}=\mathsf{0}, t=0)$ all the harmonic components are in phase, i.e.

$$\varphi_n(0) = 0 \quad n = 1, ...N, \tag{13}$$

this condition assures the existence of a stationary point for the nonlinear surface displacement $\eta(\mathsf{x},t)$. From Eq. (12) the spatial gradient and the partial derivative with respect to t have respectively expression as follows

$$\begin{aligned} \nabla\eta|_{\mathsf{x}=\mathsf{0},t=0} &= \frac{i}{2\pi}\sum_{n=1}^{N}\mathsf{k}_n\sqrt{\frac{\omega_n}{2g}}\,|B_n(0)|\exp\left[i\varphi_n(0)\right] + c.c. \\ \left.\frac{\partial\eta}{\partial t}\right|_{\mathsf{x}=\mathsf{0},t=0} &= \frac{1}{2\pi}\sum_{n=1}^{N}\sqrt{\frac{\omega_n}{2g}}\,\left.\frac{\partial B_n}{\partial t}\right|_{t=0} + c.c. \end{aligned} \tag{14}$$

If condition (13) is imposed, invoking the evolution equation (6) one can evaluate the time derivative $\left.\frac{\partial B_n}{\partial t}\right|_{t=0}$ as follows

$$\begin{aligned} \left.\frac{\partial B_n}{\partial t}\right|_{t=0} &= -i\omega_n\,|B_n(0)| - \\ &+ i\sum_{p,q,r} T_{npqr}\delta_{n+p-q-r}\,|B_p(0)|\,|B_q(0)|\,|B_r(0)| \end{aligned}$$

and from Eq. (14) both the spatial gradient $\nabla\eta$ and the time derivative $\frac{\partial\eta}{\partial t}$ vanish at $(\mathsf{x}=\mathsf{0}, t=0)$. Thus $\eta(\mathsf{x},t)$ has a stationary point at $(\mathsf{x}=\mathsf{0}, t=0)$ as in the linear case. In the following sufficient conditions will be given such that at the stationary point $(\mathsf{x}=\mathsf{0}, t=0)$ an absolute maximum is attained by $\eta(\mathsf{x},t)$. From Eq. (12) the free-surface amplitude at any time t at $\mathsf{x}=\mathsf{0}$ is

$$\eta|_{\mathsf{x}=\mathsf{0}}(t) = \frac{1}{\pi}\sum_{n=1}^{N}\sqrt{\frac{\omega_n}{2g}}\,|B_n(t)|\cos\left[\varphi_n(t)\right]. \tag{15}$$

Imposing the condition (13), yields the maximal amplitude H_{max} that can be attained at time $t=0$ for assigned values of $B_n(t=0)$, that is

$$H_{\mathsf{max}} = \frac{1}{\pi}\sum_{n=1}^{N}\sqrt{\frac{\omega_n}{2g}}\,|B_n(0)|. \tag{16}$$

From Eq. (15) it follows that at any time t

$$\eta|_{\mathsf{x}=\mathsf{0}}(t) \le \frac{1}{\pi}\sum_{n=1}^{N}\sqrt{\frac{\omega_n}{2g}}\,|B_n(t)|. \tag{17}$$

Thus a sufficient condition to have an absolute maximum at $(\mathsf{x}=\mathsf{0}, t=0)$, i.e. $H_{\mathsf{max}} \ge \eta(\mathsf{x}=\mathsf{0}, t) \quad \forall t$, is that H_{max} has to be greater than the right hand side of the inequality (17). This yields

$$\sum_{n=1}^{N}\sqrt{\frac{\omega_n}{2g}}\,|B_n(0)| \ge \sum_{n=1}^{N}\sqrt{\frac{\omega_n}{2g}}\,|B_n(t)| \qquad \forall t. \tag{18}$$

This inequality holds if the harmonic amplitudes $|B_n(0)|$ satisfy the following optimization problem

$$\max\sum_{n=1}^{N}\sqrt{\frac{\omega_n}{2g}}\,|B_n(0)| \tag{19}$$

subject to the constraints [see Eqs. (8) and (9)]

$$\begin{aligned} &\sum\nolimits_n \omega_n\,|B_n(0)|^2 + \frac{1}{2}\sum\nolimits_{n,p,q,r} T_{npqr}\delta_{n+p-q-r}\,\cdot \\ &\cdot\,|B_n(0)|\,|B_p(0)|\,|B_q(0)|\,|B_r(0)| = \\ &= \sum\nolimits_n \omega_n\tilde{B}_n^2 + \frac{1}{2}\sum\nolimits_{n,p,q,r} T_{npqr}\delta_{n+p-q-r}\tilde{B}_n\tilde{B}_p\tilde{B}_q\tilde{B}_r \end{aligned} \tag{20}$$

and

$$\sum\nolimits_n |B_n(0)|^2 = \sum\nolimits_n \tilde{B}_n^2 \qquad \sum\nolimits_n \mathsf{k}_n\,|B_n(0)|^2 = \sum\nolimits_n \mathsf{k}_n\tilde{B}_n^2. \tag{21}$$

Thus, the condition (13) is sufficient for the occurrence of an extreme crest in weakly nonlinear waves, provided the uniqueness of the solution of the constrained optimization problem (19). Condition (13) could also be necessary for the occurrence of an absolute maximum, but this needs to be proved.

NONLINEAR STATISTICS OF EXTREME CRESTS

The Theory of Quasi-Determinism for Gaussian Seas

The theory of quasi-determinism for the mechanics of linear wave groups was derived by Boccotti in the eighties, with two formulations. The first one (Boccotti, 1981,1982) enables us to predict what happens when a very high crest occurs in a fixed time and location (Lindgren, 1970,1972; Breitung, 1997; Sun, 1993); the second one (Boccotti, 1989,2000) gives the mechanics of the wave group when a very large crest-to-trough height occurs. The theory, which is exact to the first order in a Stokes expansion (Gaussian sea), is valid for any type of boundary conditions (for example either for waves in an undisturbed field or in reflection). The theory was then verified in the nineties with some small-scale field experiments (Boccotti et al. 1993a,1993b), both for waves in an undisturbed field and for waves interacting with structures. An alternative approach for the derivation of the quasi-determinism theory and a field verification off the Atlantic coast of the USA were proposed by proposed by Phillips et al. (1993a,1993b). The first formulation of the theory (derived only for the time domain) was also considered in Tromans et al. (1991) and renamed as 'New Wave theory'.

Based on the first formulation of the theory, Boccotti showed that, if in a Gaussian sea state it is known that a very high local maximum occurs in some location and time, this implies with high probability that a well defined wave-group generates the high local maximum. In detail, if a local wave maximum of given elevation H occurs at a time $t = 0$ at a fixed point $\mathsf{x} = \mathsf{0}$, with probability approaching 1, the surface displacement $\eta(\mathsf{x}, t)$ tends asymptotically to the deterministic form

$$\eta_{\mathsf{det}}(\mathsf{x}, t) = \frac{H}{\sigma^2}\int E(\mathsf{k})\cos(\mathsf{k}\cdot\mathsf{x}-\omega t)\,d\mathsf{k} \qquad \frac{H}{\sigma}\to\infty \tag{22}$$

as $H/\sigma \to \infty$, i.e. when the crest is very high with respect to the mean wave height. Here, $E(\mathsf{k})$ is the wave spectrum defined in Eq. (11) and $\sigma^2 = \int E(\mathsf{k})d\mathsf{k}$ is the variance of the Gaussian sea. An exceptionally high local maximum, with a very high degree of probability, is also a wave crest of its wave, since $\eta_{\mathsf{det}}(\mathsf{x}, t)$ attains its absolute maximum at $(t = 0, \mathsf{x} = \mathsf{0})$. A direct consequence is that the number of wave crests exceeding a fixed threshold b tends to coincide with the number of local wave maxima exceeding it, provided the fixed threshold is very high; which in its turns implies: the number of wave crests exceeding a very high threshold b tends to coincide with the number of b up-crossings (b_+), that is

$$\frac{N_{cr}(b;\mathcal{T})}{N_+(b;\mathcal{T})}\to 1 \qquad \frac{H}{\sigma}\to\infty.$$

Here, $N_{cr}(b;\mathcal{T})$ and $N_+(b;\mathcal{T})$ denote respectively the number of wave crests exceeding the threshold b and the number of b_+in the very large time interval $\mathcal{T}$. Since

$$N_+(b;\mathcal{T}) \sim \exp\left(-\frac{b^2}{2\sigma^2}\right)\mathcal{T}$$

the probability of exceedance of a wave crest height admits the following asymptotic expression

$$\Pr[H > b] = \frac{N_+(b;\mathcal{T})}{N_+(0;\mathcal{T})} = \exp\left(-\frac{b^2}{2\sigma^2}\right) \qquad \frac{b}{\sigma}\to\infty. \tag{23}$$

For discrete spectra

$$E(\mathsf{k})d\mathsf{k} = \frac{1}{2}a_n^2\delta(\mathsf{k}-\mathsf{k}_n) \tag{24}$$

the deterministic group (22) reduces to

$$\eta_{\mathsf{det}}(\mathsf{x}, t) = \frac{H}{\sigma^2}\sum_{j=1}^{N}\frac{1}{2}a_n^2\cos(\mathsf{k}_n\cdot\mathsf{x}-\omega_n t) \qquad \frac{H}{\sigma}\to\infty. \tag{25}$$

By comparing Eq. (25) with the linear surface displacement $\eta_L(\mathsf{x}, t)$ in Eq. (10) for $\mathsf{x} = 0$ and $t = 0$, yields

$$\tilde{B}_n = \frac{H}{2\sigma^2}\pi\left(\frac{\omega_n}{2g}\right)^{-1/2}a_n^2 \qquad \tilde{\varphi}_n = 0 \qquad n = 1, ...N. \tag{26}$$

This means that, in Gaussian seas with spectra (24), if a high crest of height H occurs at $(\mathsf{x} = 0, t = 0)$, with probability approaching one, the surface displacement $\eta_L(\mathsf{x}, t)$ tends to assume the deterministic wave form (25). Moreover, if the crest height H is very large the spectral components $\tilde{B}_n$, with probability approaching one, tend to be equal to the components defined in Eq. (26). In the limit of $H/\sigma \to \infty$ the statistics of the wave crest height follows asymptotically the Raleigh distribution (23).

The Nonlinear Crest Amplitude

Consider the case of deep water and define the dimensionless variables

$$X_n = \frac{|B_n(0)|}{H\sqrt{\frac{\omega_0}{2g}}}, \qquad \tilde{X}_n = \frac{\tilde{B}_n}{H\sqrt{\frac{\omega_0}{2g}}} \tag{27}$$

for $n = 1, ...N$. The optimization problem (19) is then rewritten as

$$\max_{(X_1,...X_N)\in\Re^N}\sum_{n=1}^{N}X_n\sqrt{w_n} \qquad X_n > 0 \tag{28}$$

and the constraints (20) and (21), are expressed in terms of the X_n variables as follows

$$\begin{aligned}\sum_{n=1}^{N}X_n^2 &= \sum_{n=1}^{N}\tilde{X}_n^2, \\ \sum_{n=1}^{N}|\mathsf{k}_n|\cos\theta_n X_n^2 &= \sum_{n=1}^{N}|\mathsf{k}_n|\cos\theta_n\ \tilde{X}_n^2, \\ \sum_{n=1}^{N}|\mathsf{k}_n|\sin\theta_n X_n^2 &= \sum_{n=1}^{N}|\mathsf{k}_n|\sin\theta_n\ \tilde{X}_n^2,\end{aligned} \tag{29}$$

and

$$\begin{aligned}&\sum_{n=1}^{N}w_n X_n^2 + \varepsilon_d^2\sum_{n,p,q,r}\tilde{T}_{npqr}X_nX_pX_qX_r = \\ &\sum_{n=1}^{N}w_n\ \tilde{X}_n^2 + \varepsilon_d^2\sum_{n,p,q,r}\tilde{T}_{npqr}\tilde{X}_n\tilde{X}_p\tilde{X}_q\tilde{X}_r.\end{aligned} \tag{30}$$

Here, $\varepsilon_d = |\mathsf{k}_d|\,H$ is the steepness of the linear wave and the dimensionless frequencies w_n are defined as $\omega_n = w_n\omega_0$, the angles θ_n refer to the y-axis and

$$\tilde{T}_{npqr} = \frac{1}{|\mathsf{k}_d|^3}T_{npqr}\delta_{n+p-q-r}$$

with $|\mathsf{k}_d|$ the wave number corresponding to the peak frequency of the linear spectrum. In the Euclidean space $\Re^N$ the constraints in Eq. (29) represent quadratic hypersurfaces whereas the constraint (30) represents a quartic hypersurface. Their intersection manifold $\mathcal{J} \in \Re^{N-4}$ is bounded since one of the hypersurfaces is a hypersphere.

The amplitude H_{max} of the extreme crest in weakly deep-water waves [see Eq. (16)] can be rewritten as

$$H_{\mathsf{max}} = (H_{\mathsf{max}})_L + (H_{\mathsf{max}})_{NL}\,.$$

Here, according to Eq. (27) the linear part $(H_{\mathsf{max}})_L$ and the nonlinear part $(H_{\mathsf{max}})_{NL}$ are defined respectively as

$$(H_{\mathsf{max}})_L = \frac{H}{\pi}\sum_{n=1}^{N}\sqrt{w_n}\ \tilde{X}_n \tag{31}$$

and

$$(H_{\mathsf{max}})_{NL} = \frac{H}{\pi}\sum_{n=1}^{N}\sqrt{w_n}\,(X_n - \tilde{X}_n). \qquad (32)$$

In the previous section, it has been shown that if a very high wave crest H occurs in Gaussian seas, with probability approaching one, the spectral coefficients $\tilde{B}_n$ tend to be equal to Eq. (26). This implies $(H_{\mathsf{max}})_L = H$. Thus, in the limit of $H/\sigma \to \infty$, the nonlinear crest height H_{max} is given by

$$H_{\mathsf{max}} = (1+\lambda)H \qquad \frac{H}{\sigma} \to \infty \qquad (33)$$

where the dimensionless parameter λ is defined as

$$\lambda = \frac{1}{\pi}\sum_{n=1}^{N}\sqrt{w_n}X_n - 1. \qquad (34)$$

Note that $\lambda > 0$ indicates self-focusing, i.e. the linear crest amplitude H increases due to third order nonlinear interaction among free harmonics, i.e. harmonics satisfying the linear dispersion relation. Third order effects due to bound harmonics, i.e. harmonics which do not satisfy the linear dispersion relation, are neglected, but second order effects due to bound harmonics are relevant. They break down the characteristic symmetry of Gaussian seas implying that higher crests are more probable than higher troughs (Arena & Fedele, 2002).

To evaluate the effects of second order bound harmonics the approach proposed by Fedele & Arena (2003) is considered. To the second order in the Stokes expansion the extreme wave crest has expression as follows

$$h = \sum_n A_n + \frac{1}{4}\sum_{n,s}A_nA_s\Gamma_{ns} \qquad (35)$$

where A_n are harmonic amplitudes when the highest crest occur and $\Gamma_{ns} = (|\mathsf{k}_n| + |\mathsf{k}_s|) - |\,|\mathsf{k}_n| - |\mathsf{k}_s|\,|$ is the second order transfer coefficient for the case of deep water. Setting $A_n = H\sqrt{\frac{\omega_0}{2g}}\frac{1}{\pi}\sqrt{w_n}X_n$, Eq. (35) transforms to

$$H_{\mathsf{max}} = (1+\lambda)H + \alpha\,|\mathsf{k}_d|\,H^2. \qquad (36)$$

Here, the coefficients λ is defined as in Eq. (34) and α is given by

$$\alpha = \frac{1}{4\pi^2}\sum_{n,s}\Gamma_{ns}\sqrt{w_n ws}X_nX_s.$$

Eq. (36) allows to evaluate the amplitude of an extreme nonlinear wave crest by taking in to account both third order effects due to free harmonics and second order effects due to bound harmonics.

The Nonlinear Probability of Exceedance of an Extreme Crest

The linear wave crest H follows asymptotically the Rayleigh distribution (23). This implies that, from Eq. (36), the probability of exceedance of the extreme nonlinear crest height H_{max} is given by

$$\Pr[H_{\mathsf{max}} > h] = \exp\left[-\frac{(1+\lambda)^2}{8\varepsilon_d^2\alpha^2}\left(1-\sqrt{1+\frac{4\varepsilon_d\alpha}{(1+\lambda)^2}\xi}\right)^2\right] \qquad (37)$$

where $\xi = \frac{h}{\sigma}$ and $\xi \sim \varepsilon_d^{-1+\nu}, \nu > 0$ when $\varepsilon_d \to 0$. The third order effects are neglected if $\lambda = 0$. Note that one can assume $\alpha \approx 1/2$ on deep water (Fedele & Arena, 2003).

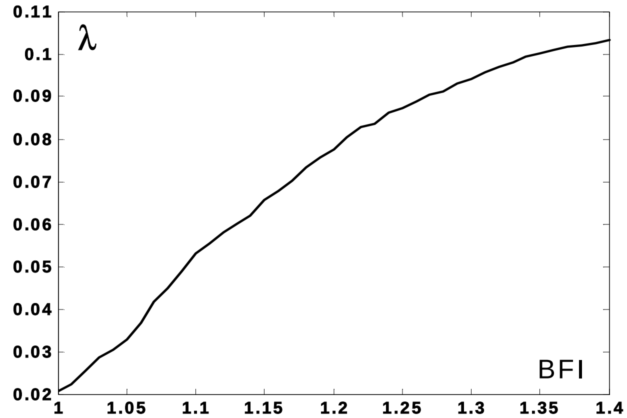

Figure 1: The parameter λ as a function of the Benjamin-Feir Index (BFI) for $\varepsilon_d = 0.05$.

NARROW-BAND SPECTRA

The optimization problem (28) will be solved for the case of unidirectional waves in deep water, i.e. $\theta_n = 0$ and $k_n = |\mathsf{k}_n|\,/\,|\mathsf{k}_d|\ \ \forall n$. Thus, in Eq. (29) the constraint relative to the wave momentum along the x direction is ignored. If the spectrum has a bandwidth ΔK, in the narrow band limit, the effects of the Benjamin-Feir instability are significant if $\Delta K/\,|\mathsf{k}_d| \leq 2\sqrt{2}\varepsilon_d$ on the time scale $t_0 \ \vee\ O(1/(\varepsilon_d^2\omega_d^2))$. Here, ω_d is the wave spectral peak frequency. Assume a narrow band spectrum where the harmonic components have dimensionless wavelengths k separated by Δk, i.e. $k_n = 1 + n\Delta k$. The corresponding dimensionless frequencies in the narrow band limit can be expressed as

$$w_n = \sqrt{1+n\Delta k} \simeq 1 + \frac{n}{2}\Delta k - \frac{n^2}{8}(\Delta k)^2 + O\left((n\Delta k)^2\right)$$

and the interaction coefficient $\tilde{T}_{npqr}$ can be assumed equal to 1 (Janseen, 2003). Then, the Eqs. (29) and (30) simplify as follows

$$\sum_{n=1}^{N}X_n^2 = \sum_{n=1}^{N}\tilde{X}_n^2, \qquad \sum_{n=1}^{N}nX_n^2 = \sum_{n=1}^{N}n\tilde{X}_n^2, \qquad (38)$$

and

$$-(\Delta k)^2\sum_{n=1}^{N}n^2X_n^2 + 8\varepsilon_d^2\sum_{n,p,q,r}X_nX_pX_qX_r \qquad (39)$$
$$= -(\Delta k)^2\sum_{n=1}^{N}n^2\,\tilde{X}_n^2 + 8\varepsilon_d^2\sum_{n,p,q,r}\tilde{X}_n\tilde{X}_p\tilde{X}_q\tilde{X}_r.$$

Janseen (2003) defines the Benjamin-Feir index (BFI)

$$BFI = \frac{2\sqrt{2}\,\varepsilon_d}{\Delta K/\left|\mathsf{k}_d\right|}.$$

in order to characterize the nonlinear behavior of the random field. If $BFI > 1$, as the frequency components of an initial wave packet change in time, energy flows from the central mode to the side-band modes. The energy eventually returns, restoring the wave to its initial state. This energy exchange occurs in time almost periodically and produce an effect of intermittence to the surface displacement: high crests occur intermittently in time, affecting the statistics of the wave crests which tends to deviate from being Gaussian (Onorato et al., 2001). Extreme events become more probable because of the Fermi-Pasta Ulam recurrence and the kurtosis of the wave distribution increases. Consider an initial wave spectrum with Gaussian shape

$$E(k) = \frac{1}{\sqrt{2\pi\sigma_k^2}} \exp\left[-\frac{(k-1)^2}{2\sigma_k^2}\right]. \tag{40}$$

The dimensionless bandwidth of this spectrum is assumed to be equal to the relative width at the energy level of one half of the spectrum maximum, i.e. $\Delta K/\left|\mathsf{k}_d\right| = \sigma_k\sqrt{2\log 2}$. Assume a characteristic steepness $\varepsilon_d = 0.05$. If one neglects the effects to due second order bound harmonics, taking the limit of $\alpha \to 0$ in Eq. (37), the probability of exceedance reduces to the form

$$\Pr[H_{\mathsf{max}} > h] = \exp\left[-\frac{\xi^2}{2(1+\lambda)^2}\right]. \tag{41}$$

The optimization problem (28) with the constraints (38) and (39) is then solved for different values of $BFI \in [1, 1.4]$. The parameter λ is plotted in fig. (1) as a function of BFI . Note that λ increases as the BFI increases implying that the probability of occurrence of extreme crests increases. In the limit of narrow-band spectrum one expects λ uniquely defined by the Benjamin-Feir index. Further studies are needed in order to determine the correct scaling for λ. Monte Carlo simulations of the Zakharov equation (where it is assumed $\tilde{T}_{npqr} \simeq 1$ for the case of narrow-band spectrum) have been performed for two different values of the BFI equal respectively to 0.9 and 1.2. The empirical crest distributions agree very well with the analytical distribution (41) as one can see from the plots in figs. (2) and (3) for the case of $BFI = 0.9$ and $BFI = 1.15$ respectively. Observe that, as the BFI increases so does the deviation from the Rayleigh distribution. Moreover the spectrum of the wave when the highest crest is attained broadens if compared to the initial spectrum [see fig. (4)]. This is a consequence of the modulation instability. Furthermore the spectrum broadens symmetrically because of the assumption of narrow-band spectra. If one relaxes this hyphotesis a downshift of the spectrum occurs (Trulsen et al., 2000,2003).

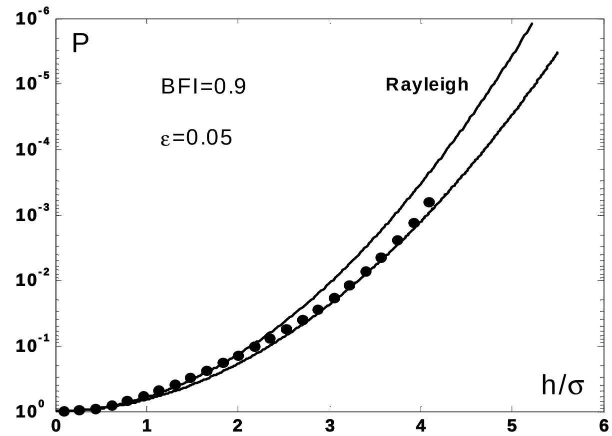

Figure 2: Comparison between the analytical probability of exceedance (41) of the wave crest (solid line) and the empirical distribution computed from Monte Carlo simulations (dotted line). Here, $BFI = 0.9$ and $\varepsilon_d = 0.05$.

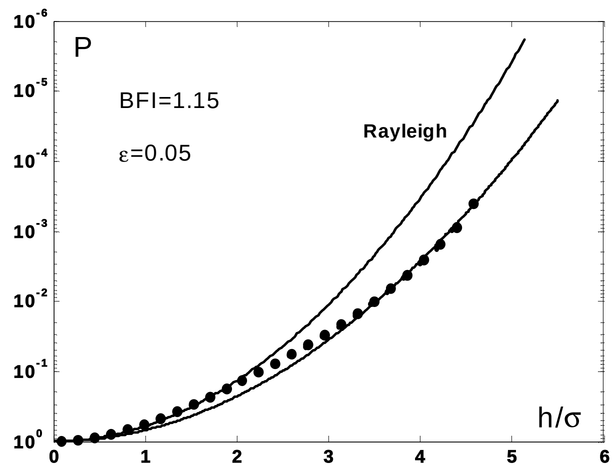

Figure 3: Comparison between the analytical probability of exceedance (41) of the wave crest (solid line) and the empirical distribution computed from Monte Carlo simulations (dotted line). Here, $BFI = 1.15$ and $\varepsilon_d = 0.05$.

JONSWAP SPECTRA

In this contest the unidirectional JONSWAP spectrum (Hasselmann et al., 1973) is adopted in the following form

$$E(k) = A\,k^{-3} \exp\left[-\frac{3}{2}k^2\right] \exp\left\{\ln\gamma \exp\left[-\frac{(\sqrt{k}-1)^2}{2\chi_2^2}\right]\right\}.$$

Here, $k = \left|\mathsf{k}\right| / \left|\mathsf{k}_d\right|$, A is the Phillips parameter, γ is the enhancement coefficient and for typical wind waves one can assume $\chi_2 = 0.08$. For $\gamma = 1$ and $A = 0.0081$ the Pierson-Moskowitz spectrum is recovered. By Taylor-expanding the spectrum around its peak one can obtain the following spectrum

$$E(k) = \frac{H_s^2}{16\pi} \frac{1}{1+(k-1)^2/\delta^2}, \qquad \delta = \sqrt{\frac{8\chi_2^2}{24\chi_2^2 + \ln\gamma}}$$

where H_s is the significant wave height and δ corresponds exactly to the half-width at half-maximum of the JONSWAP spectrum. The BFI parameter for the JONSWAP spectrum ($BFI = 2\sqrt{2}\ \varepsilon_d\ /\delta$) is plotted in fig. (5) as a function of the enhancement coefficient γ. As γ increases, the spectrum becomes higher and narrower around the spectral peak and BFI increases. Note that for $\gamma > 7$ modulation instability occurs since $BFI\ > 1$.

Consider the data of the wave elevation measured at the Draupner field in the central North Sea, during the storms in the period from December 31,1994 to January 20,1995. Joint frequency tables of successive wave crest heights and wave trough depths of the Draupner time series are provided by Wist et al. (2002) and the empirical distributions are readily obtained. The Draupner time series has significant wave height between 6.0 and 8.0 m, peak frequency $\omega_p = 0.55\ rad/s$ and mean wave period $T_m = 9.1\ s$. Consider a JONSWAP spectrum with $\gamma = 10$ ($BFI \simeq 1.1$, $\lambda \simeq 0.05$ and $\alpha \simeq 0.55$.). In Fig. 6 the probability of exceedance of the extreme crest [see Eq. (37)] is compared against the experimental distribution derived from the Draupner time series. The Rayleigh distribution and the probability of exceedance computed with $\lambda = 0$ (only second order effects) are also plotted for comparison. As one can see, the probability of exceedance (37), which consider both second and third effects, compares very well with the experimental data for high crest amplitudes, i.e. $\xi >> 1$.

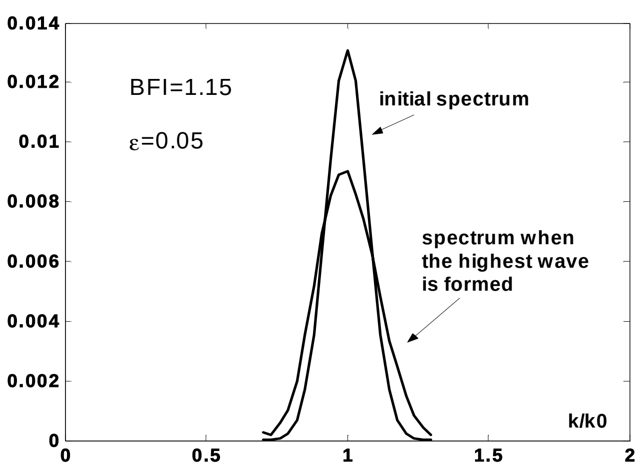

Figure 4: Spectrum at the initial stage and when the highest crest is formed ($BFI = 1.15$ and $\varepsilon_d = 0.05$)

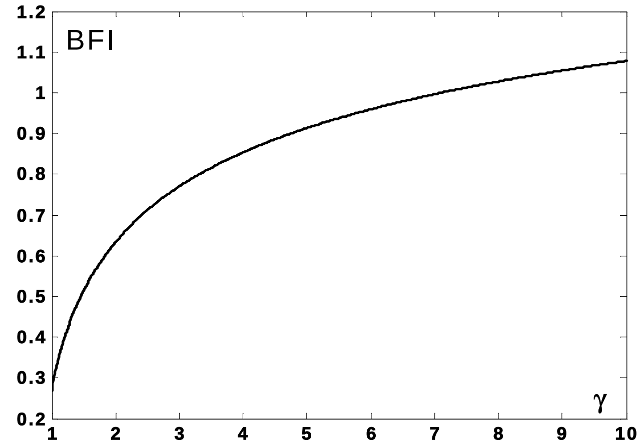

Figure 5: Benjamin-Feir index as a function of the parameter γ of the JONSWAP spectrum.

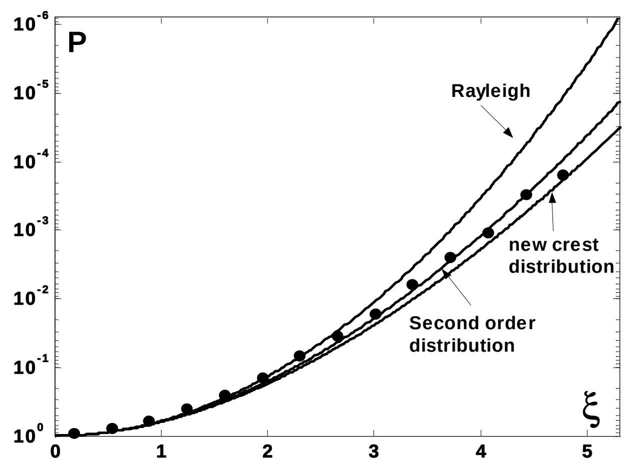

Figure 6: Probabilities of exceedance.

CONCLUSIONS

According to the Zakharov equation governing the dynamics of the spectral components of the wave envelope of the surface displacement $\eta(\mathsf{x}, t)$, the optimal spectral components which give an extreme crest at ($\mathsf{x} = 0, t = 0$) are derived. They are solutions of a well defined constrained optimization problem. By means of the theory of quasi-determinism of Boccotti, the probability of exceedance of the wave crest is then obtained. Numerical probability distributions obtained by Monte Carlo simulations of the Zakharov equation are in agreement with the new analytical distribution for the case of narrow-band spectra. Moreover the new crest distribution agrees well with the empirical distribution derived from the Draupner time series.

REFERENCES

Arena F., Fedele F. (2002) A family of narrow-band non-linear stochastic processes for the mechanics of sea waves. *Eur. J. Mech. B Fluids* 21 no. 1, 125–137.

Benney, D. J. (1962) Non-linear gravity wave interactions. *J. Fluid Mech.* 14;577–584.

Boccotti P. (1981) On the highest waves in a stationary Gaussian process. *Atti Acc. Ligure di Scienze e Lettere*;38:271-302.

Boccotti P. (1982) On ocean waves with high crests. *Meccanica*;17:16-19.

Boccotti P. (1989) On mechanics of irregular gravity waves. *Atti Acc. Naz. Lincei, Memorie*;19:11-170.

Boccotti P. (1995) A field experiment on the small-scale model of a gravity offshore platform. *Ocean Engng.* 22:615-627.

Boccotti P. (1997) A general theory of three-dimensional wave groups. *Ocean Engng.* 24:265-300.

Boccotti P. (2000) *Wave mechanics for ocean engineering.* Elsevier Science, Oxford.

Boccotti P, Barbaro G and Mannino L. (1993) A field experiment on the mechanics of irregular gravity waves. *J. Fluid Mech.* 252:173-186.

Boccotti P, Barbaro G, Fiamma V et al. (1993) An experiment at sea on the reflection of the wind waves. *Ocean Engng.* 20:493-507.

Crawford D. R., Lake B. M., Saffman P. G., Yuen H. C. (1981) Stability of weakly nonlinear deep-water waves in two and three dimensions. *J. Fluid Mech.* 105; 177–191.

Dysthe KB, Trulsen K., H. E. Krogstad & H. Socquet-Juglard (2003) Evolution of a narrow band spectrum of random surface gravity waves. *J. Fluid Mech.* 478, 1-10.

Fedele, F., Arena F. (2003) On the Statistics of High Nonlinear Random Waves. *In Proc. of the Thirteenth Inter. Offshore and Polar Eng. Conference*, Vol. III, pp. 17-22.

Hasselmann K, Barnett TP, Bouws E and al. (1973) Measurements of wind wave growth and swell decay during the Joint North Sea Wave *Project (JONSWAP). Deut. Hydrogr. Zeit.* A8:1-95.

Janssen, Peter A. E. M. (2003) Nonlinear four-wave interactions and freak waves. *J. Phys. Oceanogr.* 33, no. 4, 863–884.

Krasitskii, V. P. (1994) On reduced equations in the Hamiltonian theory of weakly nonlinear surface waves. *J. Fluid Mech.* 272, 1–20.

Krasitskii, V. P. (1990) Canonical transformations in a theory of weakly nonlinear waves with a nondecay dispersion law. *Sov. Phys. JETP* (English transl.) 71 (5), 921-927.

Komen G. J., Cavaleri L., Donelan M.,Hasselmann K., Hasselmann S.,and Janssen PAEM. (1996) Dynamics and Modelling of Ocean Waves. *Cambridge University Press*, 554 pp.

Lindgren G. (1970) Some properties of a normal process near a local maximum. *Ann. Math. Statist.* 4(6):1870-1883.

Lindgren G. (1972) Local maxima of Gaussian fields. *Ark. Mat.* 10:195–218.

Longuet-Higgins, M. S. (1976) On the nonlinear transfer of energy in the peak of a gravity-wave spectrum: a simplified model. *Trans. Roy. Soc. London Ser. A 347*, 311–328.

Longuet-Higgins, M. S. (1962) Resonant interactions between two trains of gravity waves. *J. Fluid Mech.* 12 1962 321–332.

Maes MA, Breitung KW. (1997) Direct Approximation of the extreme value distribution of nonhomogeneous Gaussian random fields. *J. Offshore Mech. Ar. Eng.* 119:252-256.

Onorato, M.; Osborne, A. R.; Serio, M. (2002) Extreme wave events in directional, random oceanic sea states. *Phys. Fluids* 14, no. 4, L25–L28.

Phillips, O. M. (1960) On the dynamics of unsteady gravity waves of finite amplitude. I. The elementary interactions. *J. Fluid Mech.* 9;193–217.

Phillips, O. M. (1961) On the dynamics of unsteady gravity waves of finite amplitude. II. Local properties of a random wave field. *J. Fluid Mech.* 11;143–155.

Phillips OM, Gu D and Donelan M. (1993) On the expected structure of extreme waves in a Gaussian sea, I. Theory and SWADE buoy measurements. *J. Phys. Oceanogr.* 23:992-1000.

Phillips OM, Gu D and Walsh EJ. (1993) On the expected structure of extreme waves in a Gaussian sea, II. SWADE scanning radar altimeter measurements. *J. Phys. Oceanogr.* 23:2297-2309.

Sun J. (1993) Tail probabilities of the maxima of Gaussian random fields. *The Annals of Probability* 21(1):34-71.

K. Trulsen, I. Kliakhandler, K. B. Dysthe & M. G. Velarde (2000) On weakly nonlinear modulation of waves on deep water. *Phys. Fluids* 12, 2432-2437.

Tromans PS, Anaturk AR and Hagemeijer P. (1991) A new model for the kinematics of large ocean waves - application as a design wave -. *Shell International Research* publ. 1042.

Onorato M, Osborne AR, Serio M and Bertone S. (2001) Freak waves in Random Oceanic Sea States *Phys. Review Letters* 86, no. 25, 5831–5834.

Wist HT, Myrhaug D, Rue H. (2002) Joint distributions of Successive wave crest heights and Successive wave trough depths for second-order nonlinear waves. *Journal of Ship Research*;46(3):175-185.

White, B. S., Fornberg B (1998) On the chance of freak waves at sea. *J. Fluid Mech.* 355, 113–138.