Localized-adjoint-finite-element-method for sub-grid stabilization of convection-dominated transport on a triangular mesh

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The advection-diffusion equation is notoriously difficult to solve for higher Peclet number when using standard Galerkin methods. Strong oscillations occur in regions of higher gradient. In order to improve the Galerkin solution two successful stabilized methods have been considered in the last decade, which are the Streamline Upwinding Petrov Galerkin method and the residual-free bubbles method. Moreover Herrera, in the context of his algebraic theory for boundary methods, has shown that optimal schemes can be derived by using optimal test functions satisfying a local adjoint boundary value problem. In this paper we apply Herrera's approach to consider unstructured triangular meshes. In order make the residual error vanish locally at each element an adjoint integro-differential boundary-value problem has been derived and solved under the hypothesis of dominant advection by the methods of successive approximations and multiple-scale perturbation. We have applied the proposed approach to the linear and quadratic elements, thereby showing that the stabilized quadratic Galerkin elements perform better than the stabilized linear Galerkin elements. Comparison with other stabilization methods is also illustrated.

1. INTRODUCTION

The advection-diffusion equation is notoriously difficult to solve for higher Peclet numbers P_e when using standard Galerkin methods. Strong oscillations occur in regions of higher gradient. Many numerical approaches have been proposed to reduce the oscillatory behavior of the Galerkin solution. In particular, the Streamline Upwinding Petrov Galerkin method of Hughes [2] adds numerical diffusion along the streamline direction damping the oscillations. Brezzi et al.[1] have proposed a residual-free bubble method. Both the above mentioned approaches belong to the general class of stabilized methods [3]. Herrera,([5][4]), in the context of his algebraic theory for boundary methods, has shown how to choose optimal test functions to derive an optimal numerical scheme with higher order convergence. According to his theory, the optimal test function satisfies a local adjoint boundary value problem. Celia et al.[6] applied this approach for structured grids, leading to an Eulerian-based numerical scheme that is able to resolve sharp-front problems with minimal numerical oscillations. In this paper we apply Herrera's approach to consider unstructured triangular grids. We first introduce a Petrov-Galerkin formulation for the advection-diffusion equation and choose the space of the test functions such that the equation of the residual error solution is identically zero. In order to achieve this condition, an adjoint integro-differential boundary-value problem has to be satisfied locally at each triangular element of the mesh. An exact analytical solution over the entire range of Peclet numbers of the latter boundary-value problem is difficult; therefore we have applied the methods of successive approximations and multiple-scale perturbation to obtain an asymptotic solution valid for higher Peclet numbers (dominant advection). Finally some benchmark problems are considered in order to show that the stabilized quadratic Galerkin elements perform better than the stabilized linear elements.

2. THE PETROV-GALERKIN METHOD

Let us consider the advection-diffusion operator $\mathcal{L} = -\nabla \cdot (\underline{\underline{D}}\, \nabla\ ) + \vec{c} \cdot \nabla$ in a bounded domain Ω in the (x, y) space, where $\underline{\underline{D}}$ is a 2x2 diffusivity tensor and $\vec{c}$ is a divergence-free velocity field. The boundary value problem considered is the following

$$\mathcal{L}(u) = f \quad on\ \Omega \qquad u|_{\partial\Omega_1} = g \qquad \left.(\underline{\underline{D}}\, \nabla u - u\vec{c}) \cdot \vec{n}\right|_{\partial\Omega_2} = r \tag{1}$$

where $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ is the exterior boundary and $g : \partial\Omega_1 \to \Re$ and $r : \partial\Omega_2 \to \Re$ as well as the source term $f : \Omega \to \Re$ are given functions. For the sake of simplicity we shall assume that Ω is a polygonal domain. We introduce on Ω a triangulation Υ_h with polygonal boundary Ω where K is the generic triangular element and $h = \max_{K \epsilon \Upsilon_h} diam(K)$. Over the entire domain Ω, we define the finite functional space

$$V_h^s = \{u \in C^s\,(\Omega)\,, u\,|_K \ \ is\ a\ polynomial\ of\ order\ s\ \ : u = g\ on\ \ \partial\Omega_1\} \tag{2}$$

Without loosing generality we choose the space of the test functions W_h as the space of the $C^0\,(K)$-continuous functions over the generic triangle K

$$W_h = \{w \in C^0\,(K) : w = 0 \quad on\ \ \partial\Omega_1\}$$

With this functional setting the Petrov-Galerkin formulation for the approximate solution $\hat{u} \in V_h^s$ is defined as the following

$$A(\hat{u}, w) + B(\hat{u}, w) = L(w) \qquad\qquad \forall w \in W_h \tag{3}$$

Where

$$\begin{gathered} A(p,q) = \textstyle\int_\Omega \left[(\underline{\underline{D}}\nabla p) \cdot \nabla q + \nabla \cdot (\vec{c}p)q\right] d\Omega \\ B(p,q) = -\textstyle\int_{\partial\Omega_2} \vec{c} \cdot \vec{n}\ pq\ dS \qquad L(q) = \int_\Omega fq\ d\Omega + \int_{\partial\Omega_2} rq\ dS \end{gathered} \tag{4}$$

The exact solution u satisfies the variational equation

$$A(u, w) + B(u, w) = L(w) \qquad\qquad \forall w \in W_h \tag{5}$$

From eqs. 3 and 5 the residual equation for $e = u - \hat{u}$ is $A(e, w) + B(e, w) = 0$ or in explicit form

$$\sum_K \int_K \mathcal{L}(e) w\ d\Omega + \sum_{K \cap \partial\Omega = \phi} \int_{\partial K} (\underline{\underline{D}}\nabla e) \cdot \vec{n}\ w dS + \int_{\partial\Omega_2} \left[(\underline{\underline{D}}\nabla e) \cdot \vec{n} - e\vec{c} \cdot \vec{n}\right] w\ dS = 0 \tag{6}$$

where in the second sum, only the internal triangular elements K are considered. Equation 6 reveals that the sum of the internal element-wise residual $\mathcal{R}(\hat{u}) = [\mathcal{L}(\hat{u}) - f] = -\mathcal{L}(e)$, the jumps of the gradient of $\hat{u}$ across the elements as well as the flux error at the boundary $\partial\Omega_2$ are in an average sense equal to zero. At this level we are free to choose the space W_h so that the residual error e vanishes. In order to do this we consider the following Green formula

$$\int_\Omega [w\mathcal{L}(e) - e\mathcal{L}^*(w)]\, d\Omega = -\int_{\partial\Omega} w(\underline{\underline{D}}\,\nabla e)\cdot\vec{n}\, dS + \tag{7}$$
$$+\int_{\partial\Omega} e[(\underline{\underline{D}}\,\nabla w)\cdot\vec{n} + w\vec{c}\cdot\vec{n}\,]dS + \sum_{K\cap\partial\Omega=\phi}\int_{\partial K}\Big[-w(\underline{\underline{D}}\,\nabla e) + e\,(\underline{\underline{D}}\,\nabla w)\Big]\cdot\vec{n}\, dS$$

where $\mathcal{L}^*(w) = -\nabla\cdot(\underline{\underline{D}}\,\nabla w) - \vec{c}\cdot\nabla w$ is the adjoint operator of $\mathcal{L}$. The formulas 6 and 7 yield the *dual residual equation* of the form

$$\sum_K\int_K e\mathcal{L}^*(w)\, d\Omega + \sum_{K\cap\partial\Omega=\phi}\int_{\partial K} e(\underline{\underline{D}}\nabla w)\cdot\vec{n}\, dS + \int_{\partial\Omega_2} e(\underline{\underline{D}}\nabla w)\cdot\vec{n}\, dS = 0 \tag{8}$$

in which we have set $e = 0$ and $w = 0$ on $\partial\Omega_1$ by definition. The residual equation 8 defines a dual velocity field w. The choice of a test function w satisfying the local adjoint equation $\mathcal{L}^*(w) = 0$, the dual flux $(\underline{\underline{D}}\,\nabla w)\cdot\vec{n}$ continuous across the internal boundaries ∂K and zero at the external boundary $\partial\Omega_2$, make the residual error to vanish. The third term in eq. 8 can be considered as the condition that the contribution from the dual flux $(\underline{\underline{D}}\,\nabla w)\cdot\vec{n}$ along the boundary ∂K vanishes in an average sense locally at each element K. The total boundary flux is $f_{tot}^{(K)} = \int_{\partial K}(\underline{\underline{D}}\nabla w)\cdot\vec{n}dS$ and can be considered as an uniform source flux distributed over the element K as $f_{tot}^{(K)}/A_K$ where A_K is the element area. We shall impose instead that the latter uniform source flux $f_{tot}^{(K)}/A_K$ in an average sense is zero, getting the new dual residual equation as

$$\sum_K\int_K e\mathcal{L}^*(w)\, d\Omega + \sum_K\int_K e\,\frac{f_{tot}^{(K)}}{A_K}\, d\Omega = 0 \tag{9}$$

Eq. 9 is identically zero if, over the triangle element K, the test function w satisfies the following integro-differential equation

$$\mathcal{L}^*(w) + \frac{1}{A_K}\int_{\partial K}(\underline{\underline{D}}\nabla w)\cdot\vec{n}\, dS = 0 \tag{10}$$

With this choice of the space W_h, the residual equation is identically zero over K for every choice of the approximate space V_h^s; hence the approximate solution $\hat{u}$ is the projection of the exact solution u onto the space V_h^s. In other words, for this choice of W_h, eq. 3 is satisfied by both the approximate solution $\hat{u}$ and the exact solution u. For the case of dominant advection, we expect that the the interfacial flux error is negligible if compared to the element residual error. In the following we shall consider only linear and quadratic elements ($s = 1, 2$); hence we shall build the test functions such that they form a basis for W_h by choosing

$$W_h(\Omega_h) = \{w \in C^0\,(K)\,, w = \phi + \delta w \;with\; \delta w\,|_{\partial K} = 0,\ \phi \in V_h^s\} \tag{11}$$

where we require that the correction δw vanishes at the boundary of the triangular element K for the completeness of the basis. Consequently we solve for the correction δw as

$$\mathcal{L}^*(\delta w) + \frac{\int_{\partial K}(\underline{\underline{D}}\nabla\delta w)\cdot\vec{n}\, dS}{A_K} = F(x,y) \qquad \delta w|_{\partial K} = 0 \tag{12}$$

where $F(x,y) = -\mathcal{L}^*(\phi) - \frac{1}{A_K}\int_{\partial K}(\underline{\underline{D}}\nabla\phi)\cdot\vec{n}\; dS$ is the generic source term which depends upon the choice of the approximation functional space V_h^s. Let us split the Petrov-Galerkin formulation 3 as follows:

$$[A(\hat{u},\phi) + B(\hat{u},\phi) - L(\phi)]_{GAL} + A(\hat{u},\delta w) + B(\hat{u},\delta w) - L(\delta w) = 0 \quad (13)$$

The first three terms in square brackets in eq. 13 represent the standard Galerkin formulation and the other terms are the sub-grid corrections. The element-wise sub-grid correction has the form

$$A(\hat{u},\delta w) = I_a(\hat{u},\delta w) + I_b(\hat{u},\delta w) \quad (14)$$

where

$$I_a = \int_\Omega \vec{c}\cdot\nabla\hat{u}\;\delta w\; d\Omega,\;\; I_b = \int_\Omega(\underline{\underline{D}}\nabla\hat{u})\cdot\nabla\delta w\; d\Omega \quad (15)$$

The added sub-grid corrections in 15 stabilize the standard Galerkin formulation. We shall particularize the expressions of the sub-grid corrections for linear and quadratic elements. For linear elements ($s = 1$) we shall show that the stabilization is enforced by adding numerical diffusion along the streamline and cross-wind directions. For dominant convection the cross-wind sub-grid correction is negligible if compared to the streamline sub-grid correction and the method reduces down to the residual free-bubble method of Brezzi [1]. For quadratic elements ($s = 2$) in the case of dominant advection, the major contribution for the stabilization comes from the added numerical diffusion and dispersion along the streamline direction. In the following we assume $\underline{\underline{D}} = \epsilon\underline{\underline{I}}$ where $\underline{\underline{I}}$ is the 2x2 identity matrix and ϵ is the diffusion coefficient.

2.1. Linear elements

Let us consider the space $V_h^1(\Omega_h)$ consisting of linear polynomials over the element K. In this case $\mathcal{L}^*(\phi) = -\vec{c}\cdot\nabla\phi$ and it is constant over the element; therefore we can solve eq. 12 with unitary source as $\mathcal{L}^*(W) + \frac{1}{A_e}\epsilon\int_{\partial K}\frac{\partial W}{\partial n}\; dS = 1$ and (see appendix for further details in the solution of eq. 12)

$$\delta w = -W\left[\mathcal{L}^*(\phi) + \frac{1}{A_K}\epsilon\int_{\partial K}\frac{\partial\phi}{\partial n}\; dS\right]$$

It is an easy task to recognize the nature of the sub-grid corrections; regarding the correction I_a

$$I_a(\hat{u},\delta w) = R\left(P_e\right)\;(\vec{c}\cdot\nabla\hat{u})\;\vec{c}\cdot\nabla\phi - \epsilon\; R\left(P_e\right)\;(\vec{c}\cdot\nabla\hat{u})\frac{1}{A_K}\int_{\partial K}\frac{\partial\phi}{\partial n}\; dS \quad (16)$$

where $R\left(P_e\right) = \int_K W\; d\Omega = \frac{A_K L_0}{c}\; f_0\left(P_e\right)$ and $f_0\left(P_e\right) = -\frac{1}{P_e} + \int_0^1 t^2\coth\left(P_e\frac{t}{2}\right)dt$ with $P_e = \frac{c\, L_0}{\epsilon}$ the local Peclet number and L_0 the maximum length of the element K along the direction of the velocity. The sub-grid correction I_a adds numerical diffusion along the streamline direction. Let us observe that the second sub-grid correction in eq. 16 due to the jump fluxes is of $O(\epsilon)$ and therefore negligible for dominant convection ($\epsilon \to 0$) as we expect. The sub-grid correction I_b can be expressed as

$$I_b(\hat{u},\delta w) = \left(\frac{\epsilon}{c}\int_K\frac{\partial W}{\partial\eta}d\Omega\right)c\frac{\partial\hat{u}}{\partial\eta}\frac{\partial\phi}{\partial\xi} + O(\epsilon^2) \quad (17)$$

in which higher terms of $O(\epsilon^2)$ have been neglected and a local orthogonal system of axes ξ, η respectively parallel and perpendicular to the velocity field has been considered (see appendix). The sub-grid correction I_b adds numerical diffusion along the cross-wind direction. It results in $I_a \sim O(1)$ and $I_b \sim O(\epsilon)$; therefore for dominant advection ($\epsilon \to 0$) the contribution from the cross-wind correction is negligible as compared to the streamline correction. Furthermore let us note that as $P_e \to \infty,\ f_0 \to 1/3$. In this case the streamline correction I_a reduces to the residual-free bubble method of Brezzi et al. [1].

2.2. Quadratic elements

Let us consider the space $V_h^2(\Omega_h)$ consisting of quadratic polynomials over the triangular element K. In this case, for solving eq. 12, we consider $F(\xi,\eta) = -\mathcal{L}^*(\phi)$ neglecting the contribution from the jump fluxes. In this case $F(\xi,\eta)$ has linear variation over the triangle K and the solution of the optimal test function has expression as

$$\delta w(\xi,\eta) = \left.\frac{\partial \phi}{\partial \xi}\right|_{\xi_-,\eta} g_1(\xi,\eta) + \left.\frac{\partial^2 \phi}{\partial \xi^2}\right|_{\xi_-,\eta} g_2(\xi,\eta)$$

where the functions $g_1(\xi,\eta), g_2(\xi,\eta)$ are reported in the appendix. For this case we consider only the correction I_a which is of $O(1)$ as $\epsilon \to 0$, expressed as:

$$I_a(\hat{u},\delta w) = \int_K c\frac{\partial \hat{u}}{\partial \xi}\left.\frac{\partial \phi}{\partial \xi}\right|_{\xi_-,\eta} g_1(\xi,\eta)\, d\Omega + \beta \int_K c\frac{\partial \hat{u}}{\partial \xi}\left.\frac{\partial^2 \phi}{\partial \xi^2}\right|_{\xi_-,\eta} g_2(\xi,\eta)\, d\Omega \qquad (18)$$

The first integral in eq. 18 adds numerical diffusion along the streamline direction; the second integral represents the variational formulation of a third order derivative $\frac{\partial^3 u}{\partial \xi^3}$ and therefore it adds numerical dispersion along the wind direction. In order to understand the effect of the second sub-grid correction in eq. 18, we have introduced a generic parameter $\beta \in [0,1]$. Numerical investigation shows that optimal solutions can be obtained with $\beta = 1/3$ as we shall show in the next section.

3. BENCHMARK PROBLEMS

In order to test the stabilized linear and quadratic elements we have considered two benchmark problems. The first problem is introduced for studying a downstream boundary layer and a characteristic internal layer that propagates along the characteristic when inflow boundary conditions are discontinuous. The domain is defined as $\Omega = \{(x,y) : 0 < x < 1,\ 0 < y < 1\}$ and the velocity field is $\vec{c} = (1, 10/3)\ m/s$; the size of the mesh used is $h = 0.06\ m$ with a mean Peclet number $\bar{P}_e = 800$ $(596 < P_e < 962)$. The inflow boundary conditions are defined at $y = 0$ as $u(x, y = 0) = H(x) - H(x - 1/3)$ where $H(x)$ is the step function. The figures 1 and 2 show the numerical solutions respectively for the linear and quadratic elements. As one can see, the quadratic elements produce a sharper front with less overshoot than the linear elements. In this case the linear elements give the same solution as the residual-free bubble method. Regarding the second problem, let us consider for the domain Ω an L-shaped geometry where the velocity field is a vortex defined as $\vec{c} = (-x, y)\ m/s$; the size of the mesh used is $h = 0.058\ m$ with a mean Peclet number $\bar{P}_e = 100$ $(256 < P_e < 20)$. The inflow boundary conditions are defined at y=0 as $u(x, y = 0) = H(x - 1/2) - H(x - 1)$ where $H(x)$ is the step function. The figures 3 and 4 show that the quadratic elements produce an enhanced numerical solution with sharper fronts and minimal oscillations.

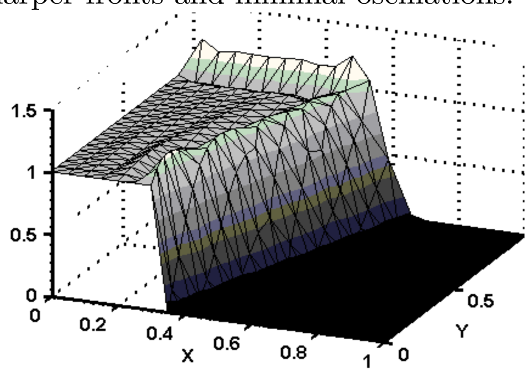

Figure1. Boundary layer - Linear FE

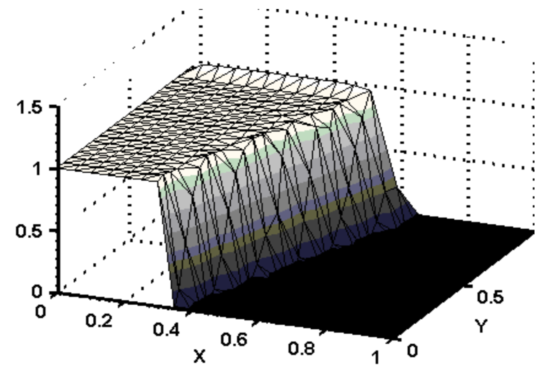

Figure2. Boundary layer - Quadratic FE

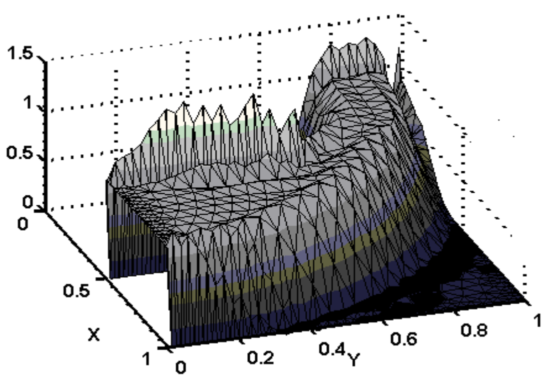

Figure3. Variable flow field - linear FE

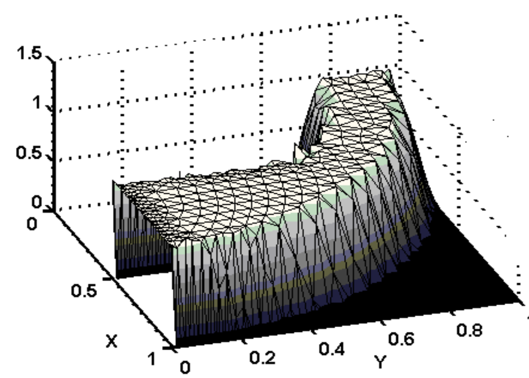

Figure4. Variable flow field - quadratic FE

4. CONCLUSIONS

We have derived a stabilization of the standard Galerkin FEM by choosing the space of the test function W_h such that the residual error equation is identically zero. The optimal test function $w \in W_h$ satisfies an adjoint integro-differential boundary value problem which is solved by the methods of successive approximations and multiple-scale perturbation, under the hypothesis of dominant convection. We recognize that the contribution from the interfacial errors is negligible for high Peclet number as one expects. Both the linear and quadratic elements have been considered. The application of the proposed approach for some benchmark cases shows that the stabilized quadratic Galerkin elements have better performance than the stabilized linear Galerkin elements.

5. APPENDIX

In order to solve the boundary value problem 12 we first refer to a local orthogonal coordinate axes system ξ,η that is parallel and perpendicular to the velocity field. Let us assume for the diffusivity tensor the form $\underline{\underline{D}} = \epsilon \underline{\underline{I}}$ where $\underline{\underline{I}}$ is the 2x2 identity matrix and ϵ is the diffusion coefficient; in this new frame we apply the method of successive approximations as

$$\mathcal{L}^*_{\xi\eta}\left(Q^{(n+1)}\right) + (1-\delta_{n0})\epsilon\frac{\int_{\partial K}\frac{\partial Q^{(n)}}{\partial n}\,dS}{A_K} = F\left(\xi,\eta\right) \qquad Q^{(n+1)}\Big|_{\partial K} = 0 \qquad (n=0,1,2,..) \tag{19}$$

where δ_{jk} is the Kronecker symbol, $\mathcal{L}^*_{\xi\eta} = -\epsilon\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right) - c\frac{\partial}{\partial\xi}$ and $\frac{\partial}{\partial n}$ are respectively the adjoint and the normal derivative operators and $c = \sqrt{c_x^2+c_y^2}$, $(c_x, c_y) = x$ and y velocities for the element K. Here $F(\xi,\eta) = -\mathcal{L}^*(\phi) - \frac{1}{A_K}\epsilon\int_{\partial K}\frac{\partial\phi}{\partial n}\,dS$. We shall solve only for the leading term $Q^{(1)}$ by applying the multiple-scale perturbation method. Let us define the inflow and outflow boundaries of the triangle element K as

$$\partial K^+ = \{\xi = \xi_+(\eta) \;:\; \vec{c}\cdot\vec{n} > 0\} \qquad\qquad \partial K^- = \{\xi = \xi_-(\eta) \;:\; \vec{c}\cdot\vec{n} < 0\}$$

As $\epsilon \to 0$, the adjoint solution $Q^{(1)}$ has a boundary layer localized at the inflow boundary ∂K^- (the velocity field is reversed for the adjoint problem) where the solution changes rapidly. In the following, for the leading solution $Q^{(1)}$, we shall drop the superscript. The diffusion flux $\epsilon\frac{\partial Q}{\partial\xi}$ and advection flux cQ are of the same order inside the boundary layer. Far from the boundary layer the advection becomes dominant and the diffusion flux can be neglected. Moreover inside the boundary layer we expect that the contribution of the cross-wind diffusion flux $\epsilon\frac{\partial Q}{\partial\eta}$ is negligible if compared to the streamline diffusion flux $\epsilon\frac{\partial Q}{\partial\xi}$, as advection becomes dominant. By defining the change of variable $\xi - \xi_-(\eta) = \epsilon\zeta$ eq. 19 (in this case $n = 0$) transforms as follows

$$-\frac{\partial^2 Q}{\partial\zeta^2} - c\frac{\partial Q}{\partial\zeta} = \epsilon F\left[\xi_-\left(\eta\right)+\epsilon\zeta,\eta\right] + \epsilon^2\frac{\partial^2 Q}{\partial\eta^2} \tag{20}$$

From eq. 20 one can recognize that the advection and streamline diffusion fluxes have the same order; furthermore the cross-wind diffusive flux $\epsilon\frac{\partial Q}{\partial\eta}$ is of order $O(\epsilon)$ compared to the streamline diffusion flux $\frac{\partial Q}{\partial\zeta}$ which is of order $O(1)$. Now we can apply the method of multiple-scale perturbation only for the variable ζ by introducing an auxiliary scale $Z = \epsilon\zeta$ (ζ is the fast scale and Z is the slow scale). We define the following pertubation expansion for Q as:

$$Q(\zeta,\eta) = Q_0\left(\zeta, Z;\eta\right) + \epsilon Q_1\left(\zeta, Z;\eta\right) + ... \tag{21}$$

For the presence of the two scales, the partial derivatives of Q with respect to the variable ζ operate as $\frac{\partial}{\partial\zeta} \to \frac{\partial}{\partial\zeta} + \epsilon\frac{\partial}{\partial Z}$, $\frac{\partial^2}{\partial\zeta^2} \to \frac{\partial^2}{\partial\zeta^2} + 2\epsilon\frac{\partial^2}{\partial\zeta\partial Z} + \epsilon^2\frac{\partial^2}{\partial Z^2}$. Plugging eq. 21 into eq. 20 one gets

$$-\left(\frac{\partial^2 Q_0}{\partial\zeta^2} + 2\epsilon\frac{\partial^2 Q_0}{\partial\zeta\partial Z} + \epsilon\frac{\partial^2 Q_1}{\partial\zeta^2}\right) - c\left(\frac{\partial Q_0}{\partial\zeta} + \epsilon\frac{\partial Q_0}{\partial Z} + \epsilon\frac{\partial Q_1}{\partial\zeta}\right) = \epsilon\left(F\big|_{\xi_-,\eta} + \frac{\partial F}{\partial\xi}\Big|_{\xi_-,\eta} Z\right) + O(\epsilon^2)$$

We therefore obtain to $O(\epsilon)$, the following hierarchy of perturbation equations :

$$O(1) \qquad \mathcal{L}_\zeta(Q_0) = 0 \qquad\qquad O(\epsilon) \qquad \mathcal{L}_\zeta(Q_1) = S_1(\zeta, Z, \eta) \tag{22}$$

where $\mathcal{L}_\zeta = -\frac{\partial^2}{\partial\zeta^2} - c\frac{\partial}{\partial\zeta}$, $S_1(\zeta, Z,\eta) = 2\frac{\partial^2 Q_0}{\partial\zeta\partial Z} + c\frac{\partial Q_0}{\partial Z} + F\big|_{\xi_-,\eta} + \frac{\partial F}{\partial\xi}\Big|_{\xi_-,\eta} Z$ and the zero boundary condition has to be satisfied by all the perturbational terms. The solution of the leading term Q_0 (see eq. 22) is

$$Q_0(\zeta, Z,\eta) = A_0(Z,\eta)z_0\left(\zeta\right) + B_0(Z,\eta)z_1\left(\zeta\right) \tag{23}$$

where $z_0(\zeta) = 1$, $z_1(\zeta) = \exp[-c\zeta]$ are the fundamental solutions of $\mathcal{L}_\zeta(z) = 0$ and $A_0(Z,\eta)$, $B_0(Z,\eta)$ are undetermined functions. Now let us solve for the Q_1 term. Plugging eq. 23 into the expression of the source term S_1 one gets

$$S_1(\zeta, Z, \eta) = H_0(Z,\eta)\, z_0(\zeta) + H_1(Z,\eta)\, z_1(\zeta)$$

Where $H_0(Z,\eta) = c\frac{\partial A_0}{\partial Z} + F|_{\xi_-,\eta} + \left.\frac{\partial F}{\partial \xi}\right|_{\xi_-,\eta} Z$, $H_1(Z,\eta) = -c\frac{\partial B_0}{\partial Z}$. The source term S_1 contains resonant forcing terms because it is a combination of the fundamental solutions $z_0(\zeta)$ and $z_1(\zeta)$. Therefore Q_1 admits particular solutions of the form $\zeta z_0(\zeta)$ and $\zeta z_1(\zeta)$ which are not admissible for the boundary layer. Thus in order to avoid non physical solutions, one has to impose the vanishing of the components of S_1 proportional to $z_0(\zeta)$ and $z_1(\zeta)$. Proceeding in this way, one gets two equations to solve for A_0,B_0 which are $H_0(Z,\eta) = 0$ and $H_1(Z,\eta) = 0$. By imposing zero values at the boundary ∂K as $Q_0(\xi_-,\eta) = 0$ and $Q_0(\xi_+,\eta) = 0$, the $O(1)$ solution Q_0 as function of the ξ,η coordinates is

$$Q_0(\xi,\eta) = \frac{1}{c}\, F|_{\xi_-(\eta),\eta}\, g_1(\xi,\eta) + \frac{1}{c} \left.\frac{\partial F}{\partial \xi}\right|_{\xi_-(\eta),\eta} g_2(\xi,\eta) \tag{24}$$

where

$$g_1(\xi,\eta) = -\left(\xi - \xi_-\right) + L(\eta)\ P(\xi,\eta) \qquad g_2(\xi,\eta) = -\frac{\left(\xi-\xi_-\right)^2}{2} + \frac{L(\eta)^2}{2}\ P(\xi,\eta)$$

with $L(\eta) = \xi_+(\eta) - \xi_-(\eta)$ and $P(\xi,\eta) = \left(\exp\left[-\frac{c\left(\xi-\xi_-\right)}{\epsilon}\right] - 1\right)/\left(\exp\left[-\frac{cL(\eta)}{\epsilon}\right] - 1\right)$. One could continue with deriving the higher order terms $Q_1, Q_2, \ldots$, but we stop at the leading term for the sake of simplicity.

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