ON THE STATISTICS OF OCEANIC WAVES

Abstract: An analytical model for the crest statistics of oceanic waves is derived based on the Breitung's asymptotic formula for the mean h-upcrossing intensity of second order narrow-band random processes. Comparisons with a wave data set collected at the Tern platform in the northern North Sea during an extreme storm are finally presented.

KEYWORDS: Crest statistics; random waves; nonlinearity; wave skewness; Tayfun distribution.

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BIOGRAPHY: Francesco Fedele received his PhD in Civil & Environmental Engineering at the University of Vermont, USA. He is an assistant professor at the School of Civil & Environmental Engineering, Georgia Institute of Technology, Savannah campus, USA. His research interests encompass practical and theoretical aspects of ocean engineering problems. These include the stochastic modeling of non-linear wave phenomena, wave group dynamics, extreme events in wave turbulence, and rogue waves. He is also interested in computational methods for biomedical engineering. In particular, he developed new tomographic algorithms for the detection of cancer in human breasts based on finite element techniques and boundary element methods.

1 Introduction

Ocean waves, to the leading order of approximation, are Gaussian and for narrow-band spectra the crest height distribution follows the Rayleigh law with probability density (Boccotti, 2000)

$$p_R(h) = \frac{h}{\sigma^2} \exp\left(-\frac{h^2}{2\sigma^2}\right), \tag{1}$$

where h is the crest amplitude and σ^2 is the variance of the sea state. In the more general case of Gaussian waves with finite-band spectra, the Rayleigh distribution serves as an upper bound for the exceedance probability of crest heights.

In reality, water waves are nonlinear, and the probability density function of the surface displacement tends to deviate from the Gaussian form. In particular, due to second order nonlinearities the water surface presents sharper crests and shallower rounded troughs. Thus, the skewness λ_3 of surface elevations is not zero (Longuet-Higgins 1963). The exact theoretical form of the corresponding distribution of nonlinear wave crests is not known under general conditions. A series expansion based on the Edgeworth's form the Gram-Charlier distribution was proposed by

Longuet-Higgins (1963). However, this leads to expressions that violate the non-negativity condition on probability densities, and crest heights of large waves can be over-predicted unrealistically in steep storm seas in deep or transitional water depths.

A convenient and simple narrow-band approximation for deep-water waves was given by Tayfun (1980, 1986, 2006) in the early eighties based upon weakly second order wave theory. This model is well known as the Tayfun distribution, and it describes the crest statistics of oceanic waves. Indeed, the recent analysis of oceanic data by Tayfun & Fedele (2007) and Fedele (2008), and the numerical simulations of Socquet-Juglard et al. (2005) both show that the Tayfun distribution explains very well the crest statistics of multidirectional random waves. Moreover, Gramstad & Trulsen (2007) provided a quantitative criterion for the minimum degree of multidirectionality for which the Tayfun distribution is a good model. Thus, for practical engineering applications where realistic oceanic conditions are characterized by multidirectional spectra, the second order Stokes theory, and thus the Tayfun model, still offers a valid theoretical framework for the wave statistics.

This paper proposes an alternative model to the Tayfun distribution that stems from an exact closed form solution of the crest distribution based on the Breitung's asymptotic formula for random processes of the average h-upcrossing intensity, h being a high threshold (Breitung and Richter 1996, Baxevani et al. 2005). The paper is structured as follows. First, the second order theory for random waves is briefly discussed in order to introduce the Tayfun model for the statistics of crest heights. Then, the Breitung's model is derived for second order narrow-band waves in deep water. Comparisons based on a wave data set collected at the Tern platform in the northern North Sea during an extreme storm are finally presented.

2 Crest statistics of second order random waves

Consider weakly nonlinear random waves propagating in deep water. The second order sea surface displacement $\zeta(\mathbf{x},t)$ from the mean sea level at a fixed point $\mathbf{x}$ in time, appropriate to long-crest deep-water waves, is given by (Tayfun 1980, 1986,2006, Fedele & Tayfun 2007, Fedele 2008)

$$\zeta = \zeta_1 + \frac{\mu}{2}\left(\zeta_1^2 - \hat{\zeta}_1^2\right), \qquad (2)$$

where $\zeta_1(\mathbf{x},t)$ is the first order Gaussian component, $\hat{\zeta}_1(\mathbf{x},t)$ is the Hilbert transform with respect to time of ζ_1 and

$$\mu = \frac{\lambda_3}{3} = \frac{\left\langle \zeta(\mathbf{x},t)^3 \right\rangle}{3\sigma^3}. \qquad (3)$$

Here, μ is the characteristic wave steepness which relates to the skewness coefficient λ_3 of the surface elevation ζ, and $\langle\cdot\rangle$ means time average. In theory, the validity of the form assumed for ζ requires that the *rms* surface gradient be sufficiently small. Specifically,

$$\mu_1 = \sqrt{\left\langle \left|\nabla\zeta_1\right|^2 \right\rangle} << 1. \qquad (4)$$

The spectral density $S(k)$ of the linear ζ_1 on the wave number space k is defined through the Fourier transform of the covariance function $\psi(T) = \langle \zeta_1(\mathbf{x}, t)\, \zeta_1(\mathbf{x}, t+T) \rangle$

$$\psi(T) = \int S(k) \cos(\omega t)\, dk,$$

where, in deep water, the wave frequency ω is a function of k through the dispersion relation $k = \omega^2/g$, with g the gravity acceleration coefficient. The spectral moments are given by $m_j = \int \omega^j S(k)\, dk$. The spectral mean frequency ω_m, the mean frequency ω_0 of zero-upcrossings of the underlying linear process ζ_1 and the bandwidth ν of the spectral density $S(\mathbf{k})$ are defined respectively as (Tayfun 1986)

$$\omega_m = \frac{m_1}{m_0}, \qquad \omega_0 = \sqrt{\frac{m_2}{m_0}}, \qquad \nu = \sqrt{\frac{m_0 m_2}{m_1^2} - 1}. \tag{5}$$

The first moment $\langle \zeta \rangle = 0$, and the higher order moments are given, correct to $O(\mu_1)$, by

$$\left\langle \zeta^2 \right\rangle = \sigma^2, \qquad \left\langle \zeta^3 \right\rangle = 3\mu, \qquad \left\langle \zeta^4 \right\rangle = 3\sigma^4,$$

where $\sigma^2 = m_0$ is the variance of ζ_1. For narrow-band waves the steepness μ can be estimated as (Tayfun 1986, 2006)

$$\mu_m = \sigma \frac{\omega_m^2}{g}. \tag{6}$$

Hereafter, the principal interest is in two-dimensional crests of the surface displacement ζ, viz. the largest maxima of a surface time series recorded at a fixed point $\mathbf{x}$. If quadratic nonlinear effects are neglected in (2), then $\zeta = \zeta_1$ is a Gaussian process of the time and the dimensionless crest height $\xi = h/\sigma$ follows the Rayleigh distribution (1) by virtue of the one-to-one correspondence between each h-upcrossing point and its nearby maximum of amplitude greater than h. If nonlinear effects are included, then the amplitude h_{nl} of the largest crest of ζ occurs whenever $\zeta_1 = h$ and $\hat{\zeta}_1 = 0$ (Tayfun 1980,1986), and it is given, from (2), in the *Tayfun form* as

$$\xi_{\max} = \xi + \frac{\mu}{2}\xi^2, \tag{7}$$

where $\xi_{\max} = h_{nl}/\sigma$ is a dimensionless amplitude. Thus, the probability of exceedance of the crest height $\xi_{\max}$ readily follows from the Rayleigh distribution of ξ as (Tayfun 1980, 1986, 2006)

$$\Pr\{\xi_{\max} > \lambda\} = \exp\left(-\frac{\xi_0^2}{2}\right), \tag{8}$$

where ξ_0 satisfies the quadratic equation (7) with $\xi_{\max} = \lambda$, that is

$$\xi_0(\lambda) = \frac{-1 + \sqrt{1 + 2\mu\lambda}}{\mu}. \tag{9}$$

Note that the Tayfun distribution (8) is an exact analytical result for the narrow-band model (2). In the following, an alternative crest distribution that stems from the Breitung's asymptotic theory (Breitung and Richter 1996) will be derived.

3 Breitung model for crest heights

Drawing on Baxevani et al. (2005), consider a fixed high threshold λ of the wave surface ζ in (2). Then, define the column vectors

$$\mathbf{p} = (p_1, p_2, ..., p_N), \qquad \mathbf{q} = (q_1, q_2, ..., q_N),$$

where $\{p_n\}$ and $\{q_n\}$ represent the sets of the spectral components of the linear surface displacement ζ_1 and its Hilbert transform respectively (see Baxevani et al. 2005 for details). The components of the vectors $\mathbf{p}, \mathbf{q}$ are defined to be independent Gaussian variables with zero mean and unit variance. Thus, the nonlinear hypersurface $\lambda = \zeta$ in the Euclidean space $\mathbb{R}^{2N}$ is given by

$$\lambda = \zeta_1(\mathbf{p}, \mathbf{q}) + \zeta_2(\mathbf{p}, \mathbf{q}), \tag{10}$$

where, for the nonlinear wave model (2)

$$\zeta_1(\mathbf{p}, \mathbf{q}) = \mathbf{z}^T \mathbf{p}, \qquad \zeta_2(\mathbf{p}, \mathbf{q}) = \frac{\mu}{2} \left(\mathbf{p}^T \mathbf{z}\mathbf{z}^T \mathbf{p} - \mathbf{q}^T \mathbf{z}\mathbf{z}^T \mathbf{q} \right).$$

Here, the column vector $\mathbf{z}$ has entries given by the spectral components $(\mathbf{z})_j = \sqrt{2S(\mathbf{k}_j)d\mathbf{k}}/\sigma$ such that $\mathbf{z}^T\mathbf{z} = 1$. As $\lambda \to \infty$, the probability of exceedance for the crest height $\xi_{\max}$ follows as

$$\Pr\{\xi_{\max} > \lambda\} = \exp\left[-\frac{\left\|\tilde{\mathbf{d}}\right\|^2}{2} \right], \tag{11}$$

where $\left\|\tilde{\mathbf{d}}\right\|$ is the minimal distance between the origin and the point $P_{\min} \in \mathbb{R}^{2N}$ identified by the column vector $\tilde{\mathbf{d}} = [\tilde{\mathbf{p}}, \tilde{\mathbf{q}}] \in \mathbb{R}^{2N}$ on the hypersurface (10). Here, $\left\|\tilde{\mathbf{d}}\right\| = \sqrt{\tilde{\mathbf{p}}^T\tilde{\mathbf{p}} + \tilde{\mathbf{q}}^T\tilde{\mathbf{q}}}$ is the classical Euclidean norm of $\tilde{\mathbf{d}}$, and T signifies the transpose. In general, the solution for $\tilde{\mathbf{d}}$ can be obtained numerically by using standard optimization techniques (Tromans and Vanderschuren, 2004).

In the following, the Lagrange multiplier method is used to derive an exact closed form solution of the point $P_{\min}$ of the hypersurface (10), pointed by $\tilde{\mathbf{d}}$, which is at the minimal distance from the origin. Indeed, consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\mathbf{p}^T\mathbf{p} + \mathbf{q}^T\mathbf{q} \right) + \chi \left[\lambda - \mathbf{z}^T\mathbf{p} - \frac{\mu}{2}\left(\mathbf{p}^T \mathbf{z}\mathbf{z}^T \mathbf{p} - \mathbf{q}^T \mathbf{z}\mathbf{z}^T \mathbf{q} \right) \right], \tag{12}$$

where the multiplier χ is introduced in order to minimize over the hypersurface (10). In appendix, it is proven that the critical vectors $(\tilde{\mathbf{p}}, \tilde{\mathbf{q}})$ that minimize $\mathcal{L}$ are given by

$$\tilde{\mathbf{p}} = \rho\xi_0\mathbf{z}, \qquad \tilde{\mathbf{q}} = \mathbf{0}, \tag{13}$$

where ξ_0 is the same as in (9), and ρ is expressed as

$$\rho = \frac{1 + \left(1 + \frac{\mu\xi_0}{2}\right)^2 \frac{\mu\xi_0}{2}}{1 + \frac{\mu\xi_0}{2}}. \tag{14}$$

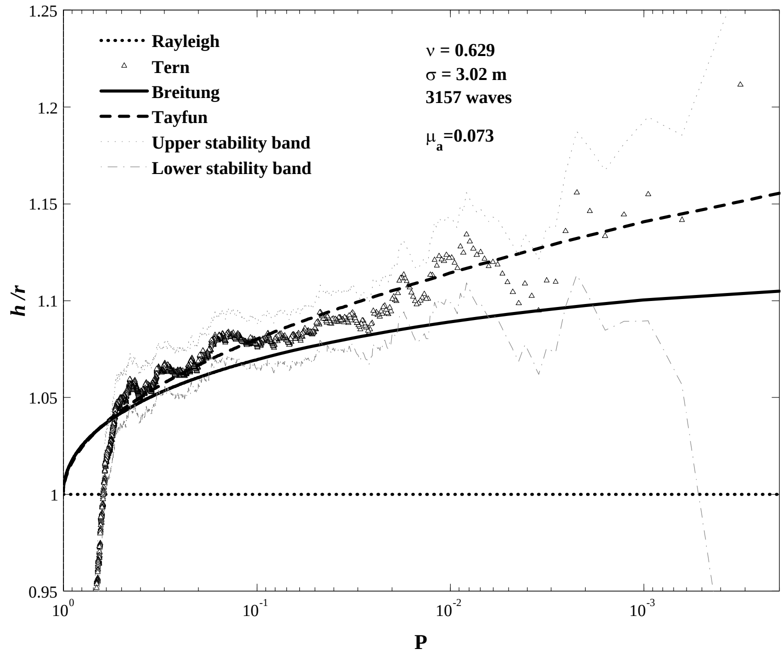

Figure 1 Ratio h/r from Tern in comparison with the Breitung and Tayfun models, respectively. Here, $r = \sigma\sqrt{-2\ln P}$ is the Rayleigh-distributed crest amplitude, and σ as the standard deviation of the sea state.

Thus, as $\lambda \to \infty$, the crest exceedance distribution follows as

$$\Pr\left\{\xi_{\max} > \lambda\right\} = \exp\left[-\frac{\tilde{\mathbf{p}}^T\tilde{\mathbf{p}} + \tilde{\mathbf{q}}^T\tilde{\mathbf{q}}}{2}\right] = \exp\left[-\frac{\xi_0^2}{2}\rho^2\right]. \tag{15}$$

Hereafter, we refer to (15) as the Breitung distribution. Note that

$$\rho = 1 + \frac{1}{2}\mu^2\xi_0^2 + O\left(\mu^3\xi_0^3\right),$$

and, correct to $O\left(\mu\xi_0\right)$, the Breitung distribution coincides with the Tayfun distribution (8) for average crest heights, because $\rho \approx 1$. For large amplitudes $\lambda >> 1$, $\mu\xi_0 >> 1$ and the Breitung model underestimates crest heights if compared to the exact Tayfun model, since $\rho > 1$. This possibly suggests that higher order terms in the Breitung's asymptotic formula of h-upcrossing intensities need to be taken into account for more accurate predictions of the crest statistics of narrow-band waves.

4 Data Comparisons

Consider the oceanic data set that comprises 9 hours of measurements gathered during a severe storm in January, 1993 with a Marex radar from the Tern platform located in the northern North Sea in 167 m water depth. The set $\{y_k\}_{k=1,n}$ contains $n = 3157$ measurements of crest heights y_k. The data, hereafter are simply referred to as Tern. The spectral properties of Tern are characterized by $\sigma = 3.02\ m$, $\nu = 0.629$ and $\lambda_3 = 0.174$ observed. To account for finite bandwidth effects, the steepness μ is estimated as $\mu_a = \mu_m \left(1 - \nu + \nu^2\right) = 0.073$ (Fedele 2008). To estimate the probability of exceeding a given crest height, we first rank-order the set $\{y_j\}_{j=1,n}$ of crest heights as $y_1 > y_2 > ... > y_n$. Then, the probability P of exceeding the threshold y_j can be estimated as

$$P = \frac{j}{n+1}, \qquad j = 1, ...n$$

with an associated error given by (Tayfun & Fedele 2007)

$$\sigma_P = \frac{1}{n+1}\sqrt{\frac{j(n-j+1)}{n+2}}.$$

Note that the largest error occurs for the largest crest height in the set because, for $j = n$, σ_P is as large as the estimate P. Clearly, the larger the sample population n, the better are the estimates. In general, the stability of estimates with negligible bias is indicated with confidence intervals. Here, the upper and lower stability bands associated with the estimate $P \approx j/(n+1)$ are defined as $P + \sigma_P$ and $P - \sigma_P$, respectively.

In figure 1, the ratio y/r of nonlinear crests y to the corresponding linear Rayleigh-distributed crests defined as $r = \sigma\sqrt{-2\ln P}$, is plotted for Tern and compared against the original Tayfun model and finally the Breitung's approximation (15). The stability bands $P + \sigma_P$ and $P - \sigma_P$ are also plotted. Clearly, the largest five estimates associated to the largest crest heights in the set are poor and thus can be neglected. It is evident that both the models seem to fit the data, but the Breitung distribution tends to slightly underestimate the data in comparison with the Tayfun model.

5 Conclusions

An exact closed form solution for the crest distribution of narrow-band deep-water waves is derived based on the Breitung's asymptotic formula of the average h-upcrossing intensity of random processes. Comparisons with oceanic measurements gathered from the Tern platform in the northern North Sea (Tern) show that the Breitung model slightly underestimates data if compared to the Tayfun model. Thus, this possibly suggests that higher order terms in the Breitung's asymptotic formula should be considered for more reliable predictions of oceanic crest heights.

6 Acknowledgements

The author thanks M. Aziz Tayfun for useful discussions and George Forristall for providing the Tern data.

7 Appendix

The gradients $\frac{\partial \mathcal{L}}{\partial \mathbf{p}}$ and $\frac{\partial \mathcal{L}}{\partial \mathbf{q}}$ of the Lagrangian (12)

$$\frac{\partial \mathcal{L}}{\partial \mathbf{p}} = \mathbf{p} - \chi \mathbf{z} - \frac{\mu}{2}\chi \; \mathbf{z} \; \mathbf{p}^T \mathbf{z}, \qquad \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{q} + \frac{\mu}{2}\chi \; \mathbf{z} \; \mathbf{q}^T \mathbf{z},$$

both vanish if the extremal vectors $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{q}}$ satisfy the following equations

$$\tilde{\mathbf{p}} = \chi \mathbf{z} + \frac{\mu}{2}\chi \; \mathbf{z} \; \tilde{\mathbf{p}}^T \mathbf{z}, \qquad \tilde{\mathbf{q}} = -\frac{\mu}{2}\chi \; \mathbf{z} \; \tilde{\mathbf{q}}^T \mathbf{z}. \tag{16}$$

These can be solved exactly by setting two new scalar parameters α and β such as

$$\alpha = \tilde{\mathbf{p}}^T \mathbf{z}, \qquad \beta = \tilde{\mathbf{q}}^T \mathbf{z}. \tag{17}$$

Then, from (16)

$$\tilde{\mathbf{p}} = \chi \left(1 + \frac{\mu\alpha}{2}\right) \mathbf{z}, \qquad \tilde{\mathbf{q}} = -\frac{\mu\beta}{2}\chi \; \mathbf{z}, \tag{18}$$

and (17) yields two linear equations for the unknowns α and β, that is

$$\alpha = \chi \left(1 + \frac{\mu\alpha}{2}\right), \qquad \beta = -\chi \frac{\mu\beta}{2}. \tag{19}$$

Non trivial solutions exist for $\mu\chi \neq 2$, and they are given by

$$\alpha = \frac{\chi}{1 - \frac{\mu\chi}{2}}, \quad \beta = 0 \qquad \text{if } \mu\chi \neq 2. \tag{20}$$

Thanks to (20), the critical vectors in (18) are explicitly given by

$$\tilde{\mathbf{p}} = \chi \left(1 + \frac{1}{2}\frac{\mu\chi}{1 - \frac{\mu\chi}{2}}\right) \mathbf{z}, \qquad \tilde{\mathbf{q}} = \mathbf{0}; \tag{21}$$

From (21) and the constraint (10), one obtains the following equation for the multiplier χ

$$\lambda = \chi \left(1 + \frac{1}{2}\frac{\mu\chi}{1 - \frac{\mu\chi}{2}}\right) + \frac{\mu}{2}\chi^2 \left(1 + \frac{1}{2}\frac{\mu\chi}{1 - \frac{\mu\chi}{2}}\right)^2. \tag{22}$$

By setting

$$w = \chi \left(1 + \frac{1}{2}\frac{\mu\chi}{1 - \frac{\mu\chi}{2}}\right), \qquad \mu\chi \neq 2,$$

(22) becomes

$$\lambda = w + \frac{\mu}{2} w^2,$$

which is identical to the quadratic equation (7) for ξ_0. Thus, $w = \xi_0$ and the multiplier χ is given by

$$\chi = \frac{\xi_0}{1 + \frac{\mu\xi_0}{2}}.$$

Note that $\mu\chi = 2$ occurs if $\mu\xi_0 = -1$. Since both ξ_0 and μ are positive, the case $\mu\chi = 2$ can be ignored without restrictions. The critical vectors are given, in the final form, by

$$\tilde{\mathbf{p}} = \rho\xi_0 \mathbf{z}, \qquad \tilde{\mathbf{q}} = \mathbf{0}, \tag{23}$$

where ρ is given in (14). The critical point $(\tilde{\mathbf{p}}, \tilde{\mathbf{q}})$ on the hypersurface (10) is at minimal distance $\beta_{\min}(\lambda) = \rho\xi_0$ from the origin. Indeed, consider a generic point

$$\mathbf{p} = \tilde{\mathbf{p}} + \mathbf{a}, \qquad \mathbf{q} = \mathbf{b}, \tag{24}$$

with $\mathbf{a}$ and $\mathbf{b}$ as arbitrary vectors. The point $(\mathbf{p}, \mathbf{q})$ is on the hypersurface (10) if $\mathbf{a}$ and $\mathbf{b}$ satisfy

$$\mathbf{z}^T\mathbf{a} + \mu\left(\rho\xi_0\mathbf{z}^T\mathbf{a} + \frac{1}{2}\mathbf{a}^T\mathbf{z}\mathbf{z}^T\mathbf{a} - \frac{1}{2}\mathbf{b}^T\mathbf{z}\mathbf{z}^T\mathbf{b}\right) = 0. \tag{25}$$

Further, the distance $\beta(\lambda)$ of the point $(\mathbf{p}, \mathbf{q})$ from the origin is given by

$$\beta^2(\lambda) = \beta^2_{\min}(\lambda) + 2\rho\xi_0\mathbf{z}^T\mathbf{a} + \mathbf{b}^T\mathbf{b} + \mathbf{a}^T\mathbf{a}.$$

The critical point $(\tilde{\mathbf{p}}, \tilde{\mathbf{q}})$ is optimal if and only if $\beta^2(\lambda) > \beta^2_{\min}(\lambda)$ and this leads to the following inequality

$$2\rho\xi_0\mathbf{z}^T\mathbf{a} + \mathbf{a}^T\mathbf{a} + \mathbf{b}^{\mathbf{T}}\mathbf{b} > 0. \tag{26}$$

that must be satisfied by any given $\mathbf{a}, \mathbf{b}$ vectors. To prove that (26) holds, define first the parameter $\kappa = \mathbf{z}^T\mathbf{a}$. If $\kappa \succeq 0$, (26) is automatically satisfied. However, κ can also be negative since, from (25), it satisfies the following quadratic equation

$$\frac{\mu}{2}\kappa^2 + (1 + \mu\rho\xi_0)\,\kappa - \frac{\mu}{2}\varrho^2 = 0, \tag{27}$$

where $\varrho = \mathbf{z}^T\mathbf{b}$; this equation admits a negative real solution given by

$$\kappa_- = \frac{-(1 + \mu\rho\xi_0) - \sqrt{(1 + \mu\rho\xi_0)^2 + \mu^2\varrho^2}}{\mu}. \tag{28}$$

For $\kappa = \kappa_-$, the inequality (26) to prove becomes

$$2\rho\xi_0\,|\kappa_-| < \mathbf{a}^T\mathbf{a} + \mathbf{b}^T\mathbf{b}. \tag{29}$$

It is easy to show that

$$2\rho\xi_0\,|\kappa_-| < |\kappa_-|^2 + \varrho^2 \tag{30}$$

holds, because substituting (28) in to (30) yields the following inequality

$$1 + (\mu\rho\xi_0)^2 + \mu^2\varrho^2 > 0,$$

which is always satisfied. Further, simple algebra shows that

$$|\kappa_-|^2 + \varrho^2 = \mathbf{a}^T\mathbf{z}\mathbf{z}^T\mathbf{a} + \mathbf{b}^T\mathbf{z}\mathbf{z}^T\mathbf{b} < \mathbf{a}^T\mathbf{a} + \mathbf{b}^T\mathbf{b} \tag{31}$$

holds for any vectors $\mathbf{a}$ and $\mathbf{b}$. Thus, from (30) and (31) the inequality (29) is always satisfied and the critical point $(\tilde{\mathbf{p}}, \tilde{\mathbf{q}})$ on the hypersurface (10) is at the minimal distance from the origin.

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