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EXTREME WAVES AND STOCHASTIC WAVE GROUPS

Francesco Fedele$^{(1)}$ **& M. Aziz Tayfun**$^{(2)}$
$^{(1)}$GEST & GMAO, NASA Goddard Space Flight Center, Greenbelt MD, USA
$^{(2)}$Dept. of Civil Engineering, College of Engineering and Petroleum, Kuwait Univ., Kuwait

ABSTRACT

We introduce the concept of stochastic wave groups to explain the occurrence of extreme waves in nonlinear random seas, according to the dynamics imposed by the Zakharov equation (Zakharov, 1999). As a corollary, a new probability of exceedance of the crest-to-trough height which takes in to account the quasi-resonance interaction is derived. Furthermore, a generalization of the Tayfun distribution (Tayfun, 1986) for the wave crest height is also proposed. The new analytical distributions explain qualitatively well the experimental results of Onorato et al. (2004,2005) and the numerical results of Juglard et al. (2005).

1 INTRODUCTION

Waves that are extremely unlikely as judged by the Raleigh distribution are called freak waves. The freak event occurred on January 1^{st} 1995 under the Draupner platform in the North Sea (Wist et al.,2002) provides evidence that such waves can occur in the open ocean. During this freak event, an extreme crest with an amplitude of 18.5 m occurred. The maximal wave height of 25.6 m was much more than twice the significant wave height of about 10.8 m. Recent studies have outlined different feasible scenarios for the occurrence of extreme waves (Kharif & Pelinivosky, 2003). In particular, one of them is based on the nonlinear mechanism of second order bound waves (Tayfun, 1980;1986). They can cause a concentration of wave energy in a small area of the ocean through the time-space focusing of a second order nonlinear wave group as explained by Fedele (2006b) and Fedele & Arena (2005) by means of the theory of quasi-determinism of Boccotti (1989,2000). According to this model the wave crest amplitude H is distributed according to the Tayfun distribution (Tayfun, 1980;1986) and a freak wave, that is a wave for which $2H/H_s > 2.2$, is a rare realization of a second order nonlinear wave population, H_s being the significant wave height. Moreover the underlying stochastic process is non-gaussian but stationary and ergodic.

Another relevant scenario is based on the third order four-wave resonance interaction of free waves (Benney, 1962, Komen et al., 1996; Janseen, 2003). In this case the weakly nonlinear energy transfer among resonant quartets occurs according to the deterministic Zakharov integraldifferential equation (Zakharov, 1999; Krasitskii, 1996). For the case of narrow-band long-crested waves the Zakharov equation reduces to the nonlinear Schrödinger (NLS) equation (Janseen, 2003) valid for narrow-band spectra or to the enhanced NLS equation derived by Dysthe (Trulsen et al., 2000), valid for broader spectral bandwidth and larger steepness.

An initial wave packet changes in time when the energy flows from the central mode to the side-band modes because of the Benjamin-Feir instability (Benjamin & Feir, 1967). If only a discrete but finite set of side-band modes are considered and if the discretization is done consistently to mantain the Hamiltonian integrability of the NLS equation (Ablowitz & Herbst, 1990), then the energy eventually flows back to the central mode restoring the wave to its initial state. This energy exchange occurs in time recurrently and it produces an effect of intermittence to the surface displacement: high crests occur intermittently in time, affecting the statistics of the wave crests which tends to deviate from being Gaussian (Janseen, 2003). Extreme events become more probable due to the Fermi-Pasta Ulam recurrence and the kurtosis of the wave distribution increases (Onorato et al. 2001; Janseen, 2003). In the limit of an infinite set of side-band modes (continuous case) the recurrence phenomenon is suppressed by phase mixing and the spectrum asymptotically relaxes toward a statistical non-gaussian steady state. Such an asymptotic behavior has been confirmed by the numerical simulations of Dysthe et al., (2003) based on an accurate discretization of the enhanced NLS equation of Dysthe (Trulsen et al., 2000). According to this second scenario a freak wave is thus a typical realization of a special wave population.

In recent wave tank experiments Onorato et. al. (2005) show that a Benjamin- Feir type modulation instability is dominant only in long-crested narrow-band waves. To characterize the nonlinear behavior of the random field, they considered the Benjamin-Feir index (BFI) introduced by Janseen (2003) but defined for the first time by Onorato et al. (2001), that is

$$BFI = \frac{\sqrt{2}\varepsilon_d}{2\Delta K / |\mathbf{k}_d|}.$$

Here, ε_d is the characteristic steepness of the linear waves, $|\mathbf{k}_d|$ is the wave number corresponding to the peak of the linear spectrum and $2\Delta K$ is the bandwidth of the wave spectrum.They investigated the spatial evolution of quasi-stationary gaussian initial conditions generated by the wavemaker and found that the kurtosis tends to exceed its gaussian value and stabilizes monotonically as the distance from the wavemaker increases. The deviation from gaussianity strongly affects the wave-crest amplitudes whose sample distribution derived from the tank measurements seems to deviate from the Tayfun distribution (Tayfun, 1980;1986). Strong deviations from the Rayleigh law were also found for the crest-to-trough height distribution (Onorato et al., 2004). Socquet-Juglard et al. (2005) arrive at the same conclusions by studying the time evolution of homogenous random fields by means of numerical simulations.

Both the experimental results of Onorato et al. (2004,2005) and the numerical simulations of Juglard et al. (2005) show that for the case of multi-directional random waves the nonlinear effects due to bound waves are dominant with respect to the four-wave resonance interaction of free waves and the Tayfun distribution explains very well the crest statistics. However, no analytical models like the Tayfun distribution are currently available for the case of third order nonlinear narrow-band waves.

The aim of this paper is to introduce a stochastic theory of wave groups to explain the occurrence of extreme waves in nonlinear random seas and their statistics. The starting point is the Zakharov equation which governs the dynamics of weakly nonlinear surface waves. Guided by the theory of quasi-determinism of Boccotti (1989,2000) and supported by the analytical work of Lindgren (1970,1972) and the regression approximation method of Rychlik (1987),we introduce the concept of the *stochastic wave group*, key element in explaining the occurrence of extreme waves either during the spatial evolution of stationary, gaussian initial conditions as in channel experiments or during the time evolution of initial homogenous gaussian random fields. As a corollary a new probability of exceedance of the crest-to-trough height which takes in to account the quasi-resonance interaction is derived. The new wave height distribution explains the strong deviations from the Rayleigh law and tends to stabilize at times (distances) larger then the Benjamin-Feir time (length) scale in agreement with the experimental results of Onorato et al. (2004). The theory presented here, is also extended to consider second order bound wave nonlinearities, thus providing a generalization of the Tayfun distribution for the wave crest height.

2 WAVE GROUPS IN A GAUSSIAN SEA

Consider the general case of three dimensional Gaussian random waves and define σ as the standard deviation of the surface displacement. In the theory of quasi-determinism, Boccotti (1989,1993a,b, 2000) assumes that a large wave crest of amplitude h has been recorded at the point $\mathbf{x} = \mathbf{x}_0 = (x_0, y_0)$ at time $t = t_0$. Then he proves that as $h/\sigma \to \infty$, with probability approaching one , a well defined wave group has passed through the point $\mathbf{x} = \mathbf{x}_0$ when the apex of its development stage occurred at time $t = t_0$. If $h/\sigma \to \infty$, i.e. the crest is very high with respect to the mean crest height, then with probability approaching 1, the surface displacement η in the neighborhood of $\mathbf{x} = \mathbf{x}_0$ and $t = t_0$ is asymptotically equal to the sum of a deterministic part $\eta_{\mathrm{det}}(\mathbf{X}, T)$ of $O(h)$ and a residual random process $r(\mathbf{X}, T)$ of $O(1)$, that is

$$\eta(\mathbf{x}_0 + \mathbf{X}, t_0 + T) = \eta_{\mathrm{det}}(\mathbf{X}, T) + r(\mathbf{X}, T) \qquad (1)$$

where

$$\eta_{\mathrm{det}}(\mathbf{X}, T) = h\frac{\Psi(\mathbf{X}, T)}{\Psi(\mathbf{0}, 0)}; \qquad (2)$$

Here, $O(x)$ means '*the same order as x*', $\mathbf{X} = (X, Y)$ and $\Psi(\mathbf{X}, T)$ is the space-time covariance given by

$$\Psi(\mathbf{X}, T) = \int_0^\infty S(\mathbf{k}) \cos(\mathbf{k} \cdot \mathbf{X} - \omega T)\, d\mathbf{k}. \qquad (3)$$

Here, $S(\mathbf{k})$ is the given wave spectrum and σ^2 is the variance of the Gaussian sea. Note that $\Psi(\mathbf{0}, 0) =$

$\int S(\mathbf{k})d\mathbf{k} = \sigma^2$. The wave frequency ω is related to the wave number $k = |\mathbf{k}|$ through the linear dispersion relation $\omega^2/g = k\tanh(kd)$ with g as the acceleration due to gravity. In the limit of $h/\sigma \to \infty$, in Eq. (1) the residual $r(\mathbf{X}, T)$ is negligible if compared to the leading term, and this implies

$$\eta(\mathbf{X}, T) \simeq \eta_{\det}(\mathbf{X}, T), \qquad \text{as } h/\sigma \to \infty \tag{4}$$

Thus, an exceptionally high local maximum, with a very high degree of probability, is also the crest of its wave, since $\eta_{\det}(\mathbf{X}, T)$ attains its absolute maximum at $(T = 0, \mathbf{X} = \mathbf{0})$.

Note that $\eta_{\det}(\mathbf{X}, T)$, or equivalently Eq.(4), can be interpreted as the wave surface around a randomly chosen very large crest (Boccotti, 1989; Lindgren 1970,1972) where now the variable h is interpreted as stochastic and its probability density function $p_h(h)$ is given by

$$p_h(h)dh = \frac{EX(h)dh}{EX_+} = \exp\left(-\frac{h^2}{2\sigma^2}\right)\frac{h}{\sigma}dh, \qquad as\ \frac{h}{\sigma} \to \infty; \tag{5}$$

Here, $EX(h)dh$ is the expected number for unit time of local maxima of the surface displacement recorded in time at a fixed point at the sea whose elevation is between h and $h + dh$, and EX_+ is the expected number per unit time of zero up-crossing of the surface displacement. This model can be thought as a first order regression approximation of the wave process locally to a randomly chosen very large crest according to Rychlik (1987). Thus, the stochastic wave group in Eq. (4) can be thought as a *'gene'* for a Gaussian sea when the interest is in the dynamics of the process at high energy levels (Fedele 2005,2006b). With high probability the occurrence of extreme events in a Gaussian sea are due to the dynamics of a wave group: an isolated extreme crest event, an extreme crest-to-trough wave event or two consecutive extreme crest events occur when the same wave group is in different configurations (Fedele,2006b).

3 WAVE GROUPS IN NON-GAUSSIAN SEAS

The stochastic interpretation of the wave group $\eta_{\det}(X, T)$ given in the previous section is the key element for explaining extreme events in nonlinear random seas. Using the three fundamental invariants of the Zakharov equation (Zakharov 1999), i.e. the wave action, wave momentum and Hamiltonian, Fedele (2006a) related the initial and final time conditions of the phase-space trajectory of the Zakharov equation for the most likely dynamical path of formation of an extreme wave. Fedele argued that, in a nonlinear random sea before the extreme wave occurs, mostly likely a wave group in its first stage of development exists and nonlinearities are negligible. Thus, this wave group resembles the characteristics of a Gaussian group at some time $T = -T_0$ prior the focussing time $T = 0$ and it can be defined as in Eq. (4). According to this equation, in absence of nonlinearities, the initial Gaussian wave group would focus at time $T = 0$ with a formation of an extreme linear wave with crest amplitude h. Due to nonlinearities, at the same time $T = 0$, the wave group will focus forming an extreme crest with different amplitude h_{nl}.

To relates the nonlinear amplitude h_{nl} and the linear crest amplitude h, Fedele (2006a) defined an extremal formulation and proposed a new semi-analytical solution for the probability of exceedance of the nonlinear wave crest height h_{nl}. The relevance of this extremal formulation is highlighted by the deterministic variant of it proposed recently by Van Groesen et al. (2006a,b) for the case of the nonlinear Schrödinger equation.

In the following the extremal formulation proposed by Fedele (2006a) is briefly revisited and its exact solution is derived for the case of nonlinearly modulated long-crested narrow-band waves. As a corollary, new analytical solutions for the probability of exceedance for the crest-to-trough heights and crest heights are provided. They represent a generalization of the Rayleigh and Tayfun distribution respectively, and explain qualitatively well the experiments of Onorato et al. (2004,2005) and the numerical simulations of Juglard et al. (2005).

3.1 The extremal formulation

Consider weakly nonlinear multidirectional water waves over a finite depth d. The free-surface $\eta(\mathbf{X}, T)$ is given by

$$\eta(\mathbf{X}, T) = \frac{1}{2\pi}\sum\sqrt{\frac{\omega_n}{2g}}\,|B_n(T)|\exp\left[i\left(\mathbf{k}_n\cdot\mathbf{X} - \omega_n T + \varphi_n(T)\right)\right] + c. \tag{6}$$

where $\varphi_n(T)$ are arbitrary time-varying phase angles and the spectral component $B_n(T)$ is defined as

$$B_n(T) = |B_n(T)|\exp[i\varphi_n(T)] \qquad n = 1, ...N$$

and $\mathbf{X} = (X, Y)$ is the horizontal spatial vector. The wave frequency ω_n is related to the wave number $\mathbf{k}_n$ through the linear dispersion relation $\omega_n^2/g = |\mathbf{k}_n|\tanh(|\mathbf{k}_n|d)$. If third order nonlinear effects are considered, then the spectral components $B_n(T)$ of the wave envelope satisfy the following discrete version of the Zakharov equation (Zakharov, 1999)

$$\frac{dB_n}{dT} = -i \sum_{p,q,r} K_{npqr} \delta_{n+p-q-r} B_p^* B_q B_r \exp\left(i \Delta\omega_{npqr}\ T\right). \tag{7}$$

Here, B_n^* denotes the complex conjugate of B_n, and the kernel $K_{npqr} = K(\mathbf{k}_n, \mathbf{k}_p, \mathbf{k}_q, \mathbf{k}_r)$. is a real function of $\mathbf{k}_n, \mathbf{k}_p, \mathbf{k}_q, \mathbf{k}_r$ and it can be derived by symmetrization as described in Krasitskii (1994). The generalized Kronecker delta $\delta_{n+p-q-r}$ denotes that summation is taken over those subscripts satisfying $\mathbf{k}_n + \mathbf{k}_p = \mathbf{k}_q + \mathbf{k}_r$ and $\Delta\omega_{npqr} = \omega_n + \omega_p - \omega_q - \omega_r$. The conserved quantities of Eq. (7) are the discrete Hamiltonian

$$\mathcal{H}\left(\{B_n(T)\}\right) = \sum\nolimits_n \omega_n B_n B_n^* + \\ +\frac{1}{2} \sum_{n,p,q,r} K_{npqr} \delta_{n+p-q-r} B_n^* B_p^* B_q B_r \exp\left(i\Delta\omega_{npqr} T\right), \tag{8}$$

the wave action $\mathcal{A}$ and wave momentum $\mathcal{M}$=($\mathcal{M}_x$,$\mathcal{M}_y$), that is

$$\mathcal{A}\left(\{B_n(T)\}\right) = \sum\nolimits_n B_n B_n^*, \tag{9}$$

$$\mathcal{M}\left(\{B_n(T)\}\right) = \sum\nolimits_n \mathbf{k}_n B_n B_n^*.$$

Here, $\{B_n(T)\}$ indicates the set of the spectral components $B_n(T)$ at time T.

Consider now the stochastic linear wave group defined in Eq.(4) in its discrete form, that is

$$\eta_{\rm det}(\mathbf{X}, T) = \frac{h}{\sigma^2} \sum_{j=1}^{N} \frac{1}{2} a_n^2 \exp\left[i\left(\mathbf{k}_n \cdot \mathbf{X} - \omega_n T\right)\right] + c.c.; \tag{10}$$

In this case the wave spectrum is given by

$$S(\mathbf{k}) = \sum_{n=1}^{N} \frac{1}{2} a_n^2 \delta(\mathbf{k} - \mathbf{k}_n), \tag{11}$$

where $\{a_n\}$ is a given set of wave amplitudes and the variable h is random with a Rayleigh probability density function $p_h(h)$ given by Eq. (5).

Assume that, at some initial time $T = -T_0$, the nonlinear surface displacement $\eta(\mathbf{X}, T)$ is described by the linear group $\eta_{\rm det}(\mathbf{X}, T)$ defined in Eq. (10), that is $\eta(\mathbf{X}, -T_0) = \eta_{\rm det}(\mathbf{X}, -T_0)$ which in the Fourier domain implies

$$B_n(T = -T_0) = \tilde{B}_n \exp\left(i\tilde{\varphi}_n\right) \qquad n = 1, ...N. \tag{12}$$

where

$$\tilde{B}_n = \frac{h}{2\sigma^2} \pi \left(\frac{\omega_n}{2g}\right)^{-1/2} a_n^2 \exp\left(-i\omega_n T_0\right),$$

$$\tilde{\varphi}_n = 0 \qquad n = 1, ...N.$$

As time varies, the nonlinearities characterized by resonant and non-resonant quartet interactions modify the initial linear wave group $\eta_{\rm det}(\mathbf{X}, T)$, but during its nonlinear evolution the time invariance of the three motion integrals given by Eqs. (8-9) must hold, that is

$$\mathcal{H}\left(\{B_n\}\right) = \mathcal{H}\left(\left\{\tilde{B}_n\right\}\right), \tag{13}$$

$$\mathcal{A}\left(\{B_n\}\right) = \mathcal{A}\left(\left\{\tilde{B}_n\right\}\right),$$

$$\mathcal{M}\left(\{B_n\}\right) = \mathcal{M}\left(\left\{\tilde{B}_n\right\}\right).$$

In this setting, we seek the critical conditions that yield the largest wave at ($\mathbf{X}$ =0, $T = 0$). These are given by imposing that all the elementary waves in Eq. (6) are in phase at the focussing time, that is

$$\varphi_n(0) = 0 \quad n = 1, ...N. \tag{14}$$

This condition ensures that the nonlinear surface displacement $\eta(\mathbf{X}, T)$ admits a stationary point at $(\mathbf{X} = \mathbf{0}, T = 0)$ and the wave amplitude is given by

$$\eta(\mathbf{0}, 0) = \frac{1}{\pi} \sum \sqrt{\frac{\omega_n}{2g}} \left|B_n(0)\right| \tag{15}$$

Here, the set of harmonic coefficients $\{B_n(0)\}$ satisfying Eq. (13) can be chosen such that $\eta(\mathbf{0}, 0)$ is the absolute maximum attained by $\eta(\mathbf{X}, T)$. Define the dimensionless frequency $w_n = \omega_n/\omega_d$ and the dimensionless variables

$$X_n(T) = h\frac{|B_n(T)|}{\sqrt{\frac{\omega_d}{2g}}} \tag{16}$$

where h is highest linear crest amplitude of the linear wave group $\eta_{\det}(\mathbf{X},T)$ and ω_d the peak frequency. Note that

$$X_n(-T_0) = \tilde{X}_n = h\frac{\left|\tilde{B}_n\right|}{\sqrt{\frac{\omega_d}{2g}}}.$$

Then, the optimal set $\{B_n(0)\}$, or equivalently the set $\{X_n\}$ (hereafter X_n indicates the values $X_n(0)$ at time $T=0$), satisfies the following optimization problem

$$\max_{(X_1,\ldots X_N)\in\Re^N} \sum_{n=1}^{N} X_n\sqrt{w_n} \qquad X_n \geqslant 0 \tag{17}$$

subject to the constraints given in Eq. (13), which in terms of the X_n (or equivalently $X_n(0)$) variables are given by

$$\sum_{n=1}^{N} X_n^2 = \sum_{n=1}^{N} \tilde{X}_n^2, \qquad \sum_{n=1}^{N} |\mathbf{k}_n|\, X_n^2 = \sum_{n=1}^{N} |\mathbf{k}_n|\, \tilde{X}_n^2, \tag{18}$$

and

$$\sum_{n=1}^{N} w_n X_n^2 + \frac{1}{2}\left(\varepsilon_d\xi\right)^2 \sum_{n,p,q,r} \tilde{K}_{npqr} X_n X_p X_q X_r =$$
$$\sum_{n=1}^{N} w_n\ \tilde{X}_n^2 + \frac{1}{2}\left(\varepsilon_d\right)^2 \cdot$$
$$\sum_{n,p,q,r} \tilde{K}_{npqr}\delta_{n+p-q-r}\tilde{X}_n\tilde{X}_p\tilde{X}_q\tilde{X}_r\ \exp\left(-i\tilde{\Delta}_{npqr}\ \omega_d T_0\right).$$

Here, $\varepsilon_d = |\mathbf{k}_d|\,\sigma$ is the characteristic steepness of the linear waves, $\xi = h/\sigma$ is the dimensionless linear wave crest amplitude, $\tilde{\Delta}_{npqr} = w_n + w_p - w_q - w_r$, and $\tilde{K}_{npqr} = \frac{1}{|\mathbf{k}_d|^3}K_{npqr}$ with $|\mathbf{k}_d|$ and ω_d respectively the wave number and frequency corresponding to the peak of the linear spectrum.

In the Euclidean space $\mathbb{R}^{\mathrm{N}}$ with N the number of harmonic components, the constraints in Eq. (13) represent four hypersurfaces. Their intersection manifold $\Gamma \in \mathbb{R}^{\text{N-4}}$ is bounded since one of the hypersurfaces is a hypersphere. Because the objective function in Eq. (17) is linear in the X_n variables, then the point solution of the optimization problem lies on the intersection manifold Γ.

The solution of the constrained optimization problem in Eq. (17) provides the relation between the highest linear amplitude h and the highest nonlinear amplitude h_{nl}, that is

$$\xi_{nl} = \lambda\left(\xi,\varepsilon_d,T_0\right)\xi \qquad \xi\to\infty \tag{19}$$

where $\xi_{nl} = h_{nl}/\sigma$ is the nonlinear dimensionless crest amplitude and the dimensionless parameter $\lambda\left(\xi,\varepsilon_d,T_0\right)$ is given by

$$\lambda\left(\xi,\varepsilon_d,T_0\right) = \frac{1}{\pi}\sum_{n=1}^{N}\sqrt{w_n}X_n\left(\xi,\varepsilon_d,T_0\right). \tag{20}$$

The set $\{X_n\left(\xi,\varepsilon_d,T_0\right)\}$ is the solution of the optimization problem (17) and it depends upon the parameters ξ,ε_d and the initial time T_0. Note that $\lambda > 1$ indicates self-focusing, i.e. the linear crest amplitude h increases due to third order nonlinear interaction among free waves, i.e. waves satisfying the linear dispersion relation. Second order effects due to bound waves are also relevant. To include second order nonlinear contributions, consider the quadratic equation of Tayfun (1980,1986, 2006) which yields the modified crest amplitude ξ_{bnl}

$$\xi_{bnl} = \xi_{nl} + \frac{\mu}{2}\xi_{nl}^2 \tag{21}$$

where ξ_{nl} is the amplitude due to third order effects given by Eq. (19), $\mu = \frac{\lambda_3}{3}$ is the *rms* of the surface gradient and λ_3 is the swewness of the random process[1]. This yields the new definition of the nonlinear crest amplitude h_{nl} which takes into account also second order nonlinearities, that is

$$\frac{h_{nl}}{\sigma} = \lambda\left(\xi,\varepsilon_d,T_0\right)\xi + \frac{\mu}{2}\lambda\left(\xi,\varepsilon_d,T_0\right)^2\xi^2. \tag{22}$$

Although Tayfun derived equation Eq. (21) for narrowband random waves, recently Fedele & Arena (2005) showed that

[1]In the original formulation of Tayfun (2006), ξ_{nl} corresponds to the linear crest height.

the equation is rather general and valid for three dimensional random waves irrespective of the spectral bandwidth. In the limit of $\xi \to \infty$ the statistics of the linear wave crest height ξ follows asymptotically the Rayleigh distribution in Eq. (5) and from Eq (21) the probability of exceedance of the nonlinear extreme wave crest h_{nl} can be easily derived thus providing a generalization of the Tayfun distribution (Tayfun 1980,1986,2006), that is

$$\Pr\left(\frac{h_{nl}}{\sigma} > x\right) = \exp\left[-\frac{1}{2}\xi(x;\varepsilon_d,T_0)^2\right], \qquad (23)$$

where $\xi(x;\varepsilon_d,T_0)$ satisfies the nonlinear equation (22) where the left-hand side h_{nl}/σ is set equal to x. A numerical solution of the optimization problem in Eq. (17) is attempted by Fedele (2006a) where the generalized probability of exceedance in Eq. (23) has been compared against the sample distribution of the Draupner time series (Wist et al. 2002).

In the following an analytical solution of the extremal problem in Eq. (17) is presented for the case of long-crested narrow-band waves.

4 STATISTICS OF LONG-CRESTED NARROW-BAND WAVES

Consider unidirectional waves travelling along the x direction, in water of finite depth D with a narrow band spectrum of dimensionless bandwidth $2\Delta K/k_d$, k_d being the peak wave number. In this case one of the four constraints in Eq. (13) relative to the wave momentum $\mathcal{M}_y$ is always satisfied since there is no transverse motion. Moreover, the optimization problem in Eq. (17) can be solved analytically because the intersection manifold $\Gamma \in \mathbb{R}^{\text{N-3}}$ of the constraints in Eq. (13) relative to the three invariants $\mathcal{A}, \mathcal{M}_x$ and $\mathcal{H}$ reduces to a single point solution as it will be shown below. Set N as the number of elementary waves, define the dimensionless wave number k_n normalized with respect to k_d, the dimensionless frequency w_n normalized with respect to $\omega_d = \sqrt{k_d \tanh(k_d D)}$ and assume $k_n = 1 + \varkappa_n$ with $|\varkappa_n| << 1$; the Taylor expansion of the dimensionless frequency w_n up to second order terms is then given by [see the book of Mei (2000) for details]

$$w_n = \sqrt{k_n \tanh(k_n\ k_d D)} \simeq$$
$$1 + f_1(k_d D)\frac{\varkappa_n}{2} - \frac{1}{8}f_2(k_d D)\ \varkappa_n^2 + o(\varkappa_n^2);$$

Here, $o(x)$ means '*of order greater than x*' and the expression of the depth-dependent coefficients f_1 and f_2 are reported in appendix B. In deep water, that is $k_d D \to \infty$, f_1 and f_2 tends to be equal to 1. We now introduce the Gaussian spectrum

$$X_n^2(k_n,T)\Delta k_n = \frac{1}{\sqrt{2\pi\Omega^2(T)}}\exp\left(-\frac{\varkappa_n^2}{2\Omega^2(T)}\right)\Delta k_n$$
$$= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z_n^2}{2}\right)\Delta z_n$$

where we have defined

$$z_n = \frac{\varkappa_n}{\Omega_T}, \qquad \Delta z_n = \frac{\Delta k_n}{\Omega_T},$$

Ω_T being the spectral bandwidth $\Omega(T)$ at time T. The half bandwidth of the spectrum corresponding to half of the spectral peak value is given by $\Delta_T = \Omega_T\sqrt{2\ln 2}$ where $\Delta_T = \Delta(T)$. For this particular choice of the spectrum, the wave action and wave momentum in Eq. (13) are conserved always. In fact, in the limit of large number of waves, that is $N \to \infty$, and for narrowband spectra, that is $\Omega_T \to 0$, the following sums relative to the wave action $\mathcal{A}$ and momentum $\mathcal{M}_x$ (in the summations and hereafter, it is implicitely assumed that the index runs from 1 to N)

$$\mathcal{A}(\{X_n\}) = \sum_n X_n^2 = \sum_n \frac{\exp\left(-\frac{\varkappa_n^2}{2\Omega_T^2}\right)}{\sqrt{2\pi\Omega_T^2}}\Delta k_n$$

$$\mathcal{M}_x(\{B_n\}) = \sum_n k_n X_n^2 = \sum_n \frac{\exp\left(-\frac{\varkappa_n^2}{2\Omega_T^2}\right)}{\sqrt{2\pi\Omega_T^2}}(1+\varkappa_n)\Delta k_n$$

both represent the Riemann sums of the two following integrals, that is

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)dz = 1$$

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)(1+\Omega_T\ z)\,dz = 1.$$

This implies that both $\mathcal{A}$ and $\mathcal{M}_x$ are time invariant for continuous $(N \to \infty)$ narrowband spectra $(\Omega_T \to 0)$. Note that the condition that $\Omega_T \to 0$ is important in order to avoid non physical negative wave numbers. The time invariance of the Hamiltonian $\mathcal{H}$ [see Eq. (19)] yields instead the following equation

$$\mathcal{F}(-T_0) = \mathcal{F}(0) \tag{24}$$

where we have defined the auxiliary function

$$\mathcal{F}(T) = \sum_1 w_1 X_1^2 + \frac{1}{2}(\varepsilon_d \xi)^2 \sum_{1234} \tilde{K}_{1234} X_1 X_2 X_3 X_4 =$$
$$-\frac{\Omega_T^2}{8} f_2 S_a + \frac{1}{2}\varepsilon_d^2 \ f_3 \ S_b(T) \ \xi^2 + o(\Omega_T^2)$$

Here, the depth-dependent parameter f_3 is reported in appendix B and the coefficients S_a and S_b are given by the following discrete sums

$$S_a = \sum_1 z_1^2 \ \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z_1^2}{2}\right) \Delta z_1 \tag{25}$$

and

$$S_b(T) = \sum_{1,2,3,4} \frac{1}{2\pi} \exp\left(-\frac{z_1^2+z_2^2+z_3^2+z_4^2}{4}\right) \cdot$$
$$\cdot \tilde{K}_{1234} \cos\left(\Delta w_{1234}\omega_d T\right) \delta_{1+2-3-4} \sqrt{\Delta z_1 \Delta z_2 \Delta z_3 \Delta z_4}.$$

respectively; here, each summation index runs from 1 to N, $\Delta w_{1234} = w_1 + w_2 - w_3 - w_4$ and ω_d the wave frequency corresponding to the peak of the linear spectrum. The generalized Kronecker delta $\delta_{1+2-3-4}$ denotes that summation is taken over those subscripts satisfying $k_1 + k_2 = k_3 + k_4$. Moreover, in the narrowband limit the Zakharov kernal $\tilde{K}_{1234} \rightarrow f_3(k_d D)$ and the NLS equation is recovered (Janseen, 2003, Onorato et al. 2004). The parameter f_3 tends to one as the depth increases to infinity and becomes negative for values of $k_d D < 1.363$ as one can see from Fig. (1). In such a case the modulation instability disappears and Stokes waves are stable to perturbations. In appendix it is shown that for continuous narrowband spectra, that is $N \rightarrow \infty$ and $\Omega_T \rightarrow 0$, the continuous form of $\mathcal{F}(T)$ is given by

$$\mathcal{F}(T) = -\frac{\Omega_T^2}{8} f_2 + \frac{1}{2}\beta(T)\varepsilon_d^2 \ f_3 \xi^2 \tag{26}$$

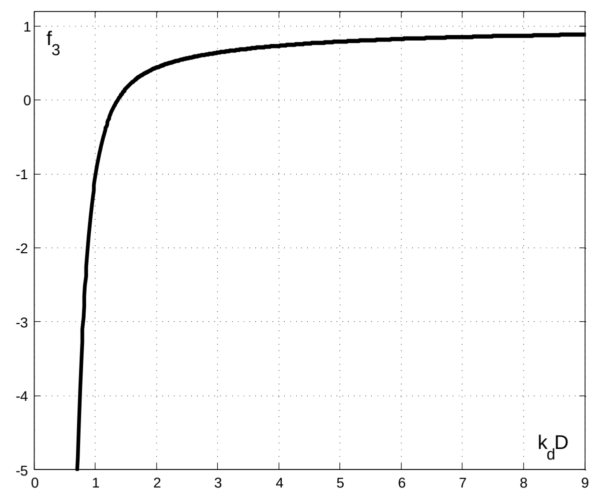

Figure 1. The parameter f_3 as function of the dimensionless $k_d D$.

where we have defined the time-varying function

$$\beta(T) = \frac{\cos\left(\frac{1}{2}\arctan \frac{1}{2\ln 2}\frac{\varepsilon_d^2}{BFI_T^2} f_3 \ \omega_d T\right)}{\sqrt[4]{\left(\frac{1}{2\ln 2}\frac{\varepsilon_d^2}{BFI_T^2} f_3 \ \omega_d T\right)^2 + 1}}. \tag{27}$$

and the BFI_T parameter at time T is given in the following orm (Onorato et al. 2004)

$$BFI_T = \frac{\sqrt{2}\varepsilon_d}{\Delta_T} f_D$$

where the depth factor f_D has expression as follows:

$$f_D = \sqrt{\frac{|f_3|}{f_2}}.$$

Note that the depth factor f_D decreases on diminuishing water depths and vanishes at $k_d D = 1.363$ as one can see from Fig.(2). As a consequence, the modulation instability diminishes its strength as the depth decreases and disappears for $k_d D = 1.363$. Thus equation (26) is valid for $k_d D \geq 1.363$. Note that the Hamiltonian invariance in Eq. (24) allows to relate only the initial and final time conditions of the phase-space trajectory of the NLS equation characterizing the most likely dynamical path of formation of an

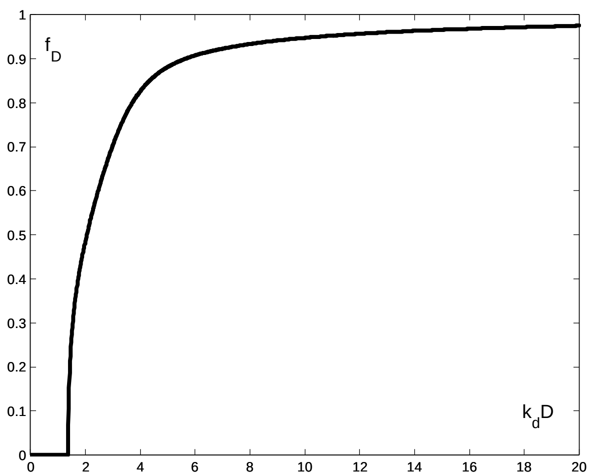

Figure 2. The depth factor f_D as function of the dimensionless depth $k_d D$.

extreme wave. In fact, Eq. (24) yields the following relation between the spectral bandwidth at the different times $T = -T_0$ and $T = 0$, that is

$$\Omega_0^2 - \Omega_{-T_0}^2 = 4\varepsilon_d^2 \; f_d \left[1 - \beta(-T_0)\right] \xi^2. \tag{28}$$

Observe that $\Omega_0 \geq \Omega_{-T_0}$, and the spectral bandwidth always increases in time since $\beta(T) \leq 1 \;\; \forall T$. The intersection manifold $\Gamma \in \mathbb{R}^{\text{N-3}}$ of the constraints relative to the three invariants $\mathcal{A}, \mathcal{M}_x$ and $\mathcal{H}$ reduces to a single point solution given by the Gaussian spectrum with bandwidth Ω_0 if Eq. (28) holds. Thus, the optimization problem in Eq. (17) admits a unique solution. In the continuous narrowband limit, that is $N \to \infty$ and $\Omega_T \to 0$, the ratio between the optimal nonlinear crest amplitude $\xi_{nl} = h_{nl}/\sigma$ and the linear crest amplitude $\xi = h/\sigma$ is given by

$$\frac{h_{nl}}{h} = \frac{\xi_{nl}}{\xi} = \frac{\sum_n \sqrt{w_n} X_n(0)}{\sum_n \sqrt{w_n} X_n(-T_0)} \simeq \sqrt{\frac{\Omega_0}{\Omega_{-T_0}}} \geq 1$$

and this ratio is always greater than or equal to 1 because $\Omega_0 \geq \Omega_{-T_0}$, implying that extreme waves are always larger than Gaussian waves. Furthermore from Eq. (28) it readily follows that

$$\frac{\xi_{nl}}{\xi} = \sqrt[4]{1 + 4\ln 2 \;\; \left[1 - \beta(-T_0)\right] \;\; BFI_{-T_0}^2 \xi^2} \tag{29}$$

which yields a relation between the nonlinear wave crest height ξ_{nl} and the linear crest height ξ in terms of the initial BFI at time $-T_0$. In simple words, the initial spectrum with bandwidth Ω_{-T_0} at time $T = -T_0$ characterizes an initial linear wave group that, as time goes on, nonlinearly evolves and its spectral bandwidth tends to broaden at $T = 0$ when the largest wave occurs at the focussing point $X = 0$.

Thus, larger waves than the Gaussian waves may form because of both the constructive interference of the elementary waves and the broadening of the wave spectrum due to the Benjamin-Feir modulation instability.

4.1 The wave crest distribution

Because the linear crest height ξ is distributed according to the Rayleigh law, from Eq. (29) the probability of exceedance of the wave crest ξ_{nl} at some time T is given by

$$\Pr\left(\xi_{nl} > y\right) = \exp\left(-\frac{\xi^2(y)}{2}\right) \tag{30}$$

where $\xi(y)$ satisfies the algebraic equation

$$y^4 = \xi^4 + \chi \; \xi^6 \tag{31}$$

and the coefficient χ is defined as follows:

$$\chi(T; \alpha, k_d D) =$$
$$4\ln 2 \; BFI^2 \left(1 - \frac{\cos\left(\frac{1}{2}\arctan \frac{1}{2\ln 2}\frac{\varepsilon_d^2}{BFI_T^2} f_3 \; \alpha\right)}{\sqrt[4]{\left(\frac{1}{2\ln 2}\frac{\varepsilon_d^2}{BFI_T^2} f_3 \; \alpha\right)^2 + 1}}\right);$$

Here, the parameter $\alpha = \omega_d T$ and the BFI is the Benjamin Feir index BFI_{-T_0} of the wave field at some initial time as in in Eq. (29). Note that the coefficient χ depends upon the time scale T because we are considering the time evolution of a spatial homogenous random field. The Eq. (30) is also valid for the case of spatial evolution of an initial stationary Gaussian state in a wave tank if one sets $\alpha = k_d X$, X being the distance from the wavemaker. In this case, BFI represents the Benjamin-Feir index of the waves generated at the wavemaker, i.e. at $X = 0$. The probability of exceedance in Eq. (30) can be interpreted as the ratio between the number of waves N_y whose crest amplitude ξ_{nl} exceeds the threshold y at time T (at distance X from the wavemaker) and the total number of waves N_w occurring in space (in time) at time T (at distance X). If the threshold y is very large, a wave with $\xi_{nl} > y$ is the central wave of a wave group that has reached its maximum at time T (at distance X from the wavemaker).

To compute the probality of exceedance in Eq. (30), we shall use the method of successive approximations to solve for the positive fix point solution ξ of Eq. (31) which satisfies the following convergent recurrent relation

$$\xi_{j+1}^2 = \frac{y^2}{\sqrt{1+\chi\xi_j^2}}, \qquad j = 0, 1, ... \tag{32}$$

where for $j = 0$ we set $\xi = 0$. To highlight certain properties of the proposed distribution (30), we solve this recurrent relation up to iteration $j = 2$ obtaining the approximation

$$\Pr\left(\xi_{nl} > y\right) \simeq \exp\left(-\frac{y^2}{2\sqrt{1+\chi\, y^2}}\right). \tag{33}$$

which is valid only for small values of χ (for real data comparisons the numerical solution of the recurrent relation in Eq. (32) is necessary). Including second order Stokes effects, from Eq. (22) one obtains the generalized Tayfun distribution for the wave crest

$$\Pr\left(\tilde{\xi}_{nl} > y\right) \simeq \exp\left(-\frac{\left(-1+\sqrt{1+2\mu\, y}\right)^2}{2\mu^2\sqrt{1+\chi\ \left(-1+\sqrt{1+2\mu\, y}\right)^2}}\right) \tag{34}$$

where $\tilde{\xi}_{nl} = \xi_{nl} + \frac{\mu}{2}\xi_{nl}^2$ is the nonlinear crest height which takes in to account also second order nonlinearities.

Define the modulation time scale $T_{bf} = 2\pi/(\omega_d\varepsilon_d^2)$ and length scale $L_{bf} = 2\pi/(k_d\varepsilon_d^2)$ respectively. Then, it is interesting to note that for $T << T_{bf}$ ($X << L_{bf}$), that is $\alpha \to 0^+$, the coefficient χ in Eq. (32) tends to zero and the crest distribution (34) tends to the Tayfun distribution (Tayfun 1986,2006), that is

$$\Pr\left(\tilde{\xi}_{nl} > y\right) \to \exp\left(-\frac{\left(-1+\sqrt{1+2\mu\, y}\right)^2}{2\mu^2}\right)$$

and larger wave crests are well described by second order Stokes theory. This means that for time scales (distances) less than the modulation time (length) scale T_{bf} (L_{bf}) the modulation instability has not developed yet and the waves follows the Tayfun distribution. For T much longer than the modulation period T_{bf} or for distances X away from the wavemaker ($X >> L_{bf}$), that is for increasing α approaching infinity, the coefficient χ tends monotonically to a constant and the new crest distribution (34) monotonically deviates from the Tayfun distribution and tends to relax toward a steady state distribution given by

$$\Pr\left(\tilde{\xi}_{nl} > y\right) \to \tag{35}$$
$$\exp\left(-\frac{\left(-1+\sqrt{1+2\mu\, y}\right)^2}{2\mu^2\sqrt{1+4\ln 2\ BFI^2\left(-1+\sqrt{1+2\mu\, y}\right)^2}}\right)$$

In this case the modulation instability is fully developed and permanently alters the initial wave field in agreement with the experimental results of Onorato et al. (2005) and the numerical simulations of Juglard et al. (2005).

4.2 The crest-to-trough height distribution

The probability of exceedance of the wave height H is easily obtained from Eq. (33) as

$$\Pr\left(H/H_s > x\right) = \exp\left(-\frac{\xi^2(x)}{2}\right) \tag{36}$$

where $H_s = 4\sigma$ is the significant wave height and $\xi(x)$ satisfies the algebraic equation (31) for $y = 2x$, that is

$$16x^4 = \xi^4 + \chi\,\xi^6 \tag{37}$$

which can be solved by means of the convergent recurrent relation

$$\xi_{j+1}^2 = \frac{16x^4}{\sqrt{1+\chi\xi_j^2}}, \qquad j = 0, 1, ... \tag{38}$$

whose solution up to iteration $j = 2$ yields the following approximation

$$\Pr\left(H/H_s > x\right) \simeq \exp\left(-\frac{2x^2}{\sqrt{1+4\chi\, x^2}}\right). \tag{39}$$

Observe that for $T << T_{bf}$ ($X << L_{bf}$) the modulation instability is weakly developing and

$$\Pr\left(H/H_s > x\right) \to \exp\left(-2\,x^2\right)$$

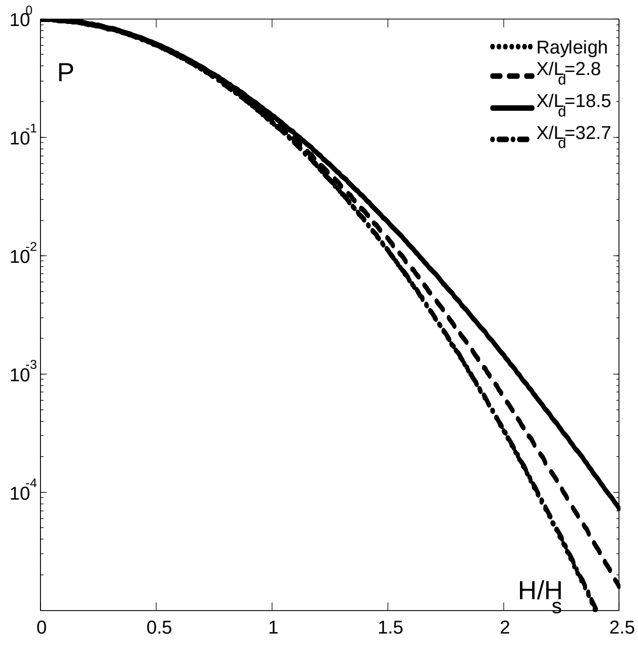

Figure 3. Wave height probabilities of exceedance for X/L_d=2.8;18.5;32.7 compared to the Rayleigh distribution ($BFI = 0.6$).

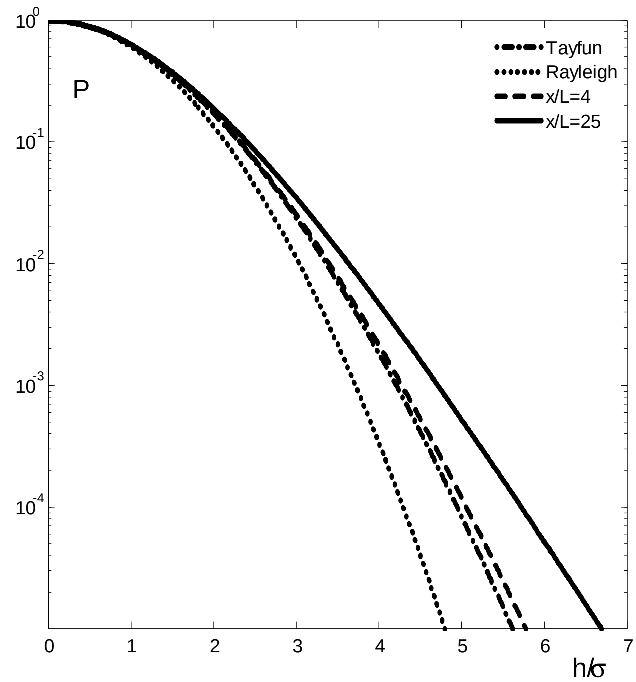

Figure 4. Wave crest distributions X/L_d=4;25 compared to both the Tayfun and Rayleigh distributions ($BFI = 0.6, \mu = 0.071$).

implying that larger wave heights follow approximately the Rayleigh law. For T much longer than the modulation period T_{bf} or for distances X away from the wavemaker ($X >> L_{bf}$), the modulation instability is fully developed and the new wave height distribution monotonically deviates from the Rayleigh law and tends toward a steady form given by

$$\Pr\left(H/H_s > x\right) \rightarrow \exp\left(-\frac{2x^2}{\sqrt{1+16\ln 2\ BFI^2\ x^2}}\right).$$

The initial wave field is permanently altered and strongly intermittent due to the increased kurtosis as shown by Onorato et al. (2004).

5 COMPARISONS

Consider the experimental results[2] of Onorato et al. (2004,2005). They investigated the case of a wave field generated at the wavemaker with $BFI = 0.6$, steepness $\varepsilon_d = 0.071$, significant wave height $H_s = 0.16m$, depth factor $f_d = 0.95$, peak period $T_d = 1.5s$ and correspondent wavelength $L_d = 3.51m$ and the Tayfun parameter is $\mu = 0.071$. Figs. 4-5-6 in Onorato et al. (2004) show the plots of the experimental sample distribution of the crest-to-trough height at different distances $X/L_d = 2.8, 18.5, 32.7$ from the wavemaker. From these figures it is clear the increasing deviations from the Rayleigh law away from the wave maker due to the developing modulation instability. The latter experimental results agree well with the analytical distribution computed using Eq. (36) for the same X/L_d values used by Onorato et al. (2004) as one can see from Fig. (3). To compute this analytical distribution, the recurrent relation in Eq.(38) has been solved numerically and its solution is exact within numerical accuracy. Note that in Fig. (3) the curve relative to $X/L_d = 2.8$ is pratically coincident with the Rayleigh distribution and X/L_d increases, deviations from the Gaussian conditions occur. The wave height distribution is not affected by second order nonlinearities, which instead can modify the wave crests. This fact is clearly shown by the experimental results pre-

[2]Onorato et al. (2004,2005) define the steepness as $2\varepsilon_d = 0.142$ and thus the correspondent value of the Benjamin Feir index is $2BFI = 1.2$.

sented by Onorato et al. (2005). From the same experiment described above ($BFI = 0.6$, $\mu = 0.071$) they showed that closer to the wave maker ($X/L_d << 1$), the Tayfun distribution explains well the experimental data, whereas away from the wavemaker ($X/L_d >> 1$) strong deviations from the Tayfun distribution occur. Solving the recurrent relation in Eq. (32), the generalized Tayfun distribution proposed in Eq. (30) is plotted in Fig. (4) for two values of X/L_d=4;25. As one can see, increasing deviations from the Tayfun distribution occurs as the distance X/L_d increases.

These analytical results qualitatively agree with the experiments of Onorato et al. (2004) and the numerical simulations of Juglard et al. (2005), thus providing evidence that the concept of the stochastic wave group is useful for explaining the occurrence of extreme event occurs in random seas due to modulation instability. More analysis and numerical simulations are needed in order to fully validate both the theory and the new analytical distributions presented here, but this will be discussed elsewhere.

6 CONCLUSIONS

Guided by the theory of quasi-determinism of Boccotti (1989,2000) and supported by the analytical work of Lindgren (1970,1972) and the regression approximation method of Rychlik (1987), the concept of stochastic wave groups is presented for explaining the occurrence of an extreme wave in third order nonlinear random seas. In particular, for the case of nonlinearly modulated narrow-band waves a new probability of exceedance of the wave height which takes in to account the quasi-resonance interaction is derived. A generalization of the Tayfun distribution for the wave crest height is also provided. The new distributions explain qualitatively well the experimental results of Onorato et al. (2004,2005) and the numerical results of Juglard et al. (2005).

7 APPENDIX A

To simplify the expression of $\mathcal{F}(T)$ given in Eq. (26) consider only the quasi-resonant interactions of the following type

$$k_1 = 1 + \varkappa_1, \quad k_2 = k_1, \quad k_3 = k_1 + \varkappa_2, \quad k_4 = k_1 - \varkappa_2$$

where $\forall \varkappa_1,\ \varkappa_2, << 1$. It follows that

$$\Delta w_{1234} = w_1 + w_2 - w_3 - w_4 = -\frac{1}{4} f_2(k_d D)\ z_2^2 \Omega_{-T_0}^2 + o(z_2^2, z_1^2)$$

where f_2 is given by in Eq. (41). Then, S_a and S_b in Eqs. (25,26) simplify respectively as follows

$$S_a = \sum_1 z_1^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z_1^2}{2}\right) \Delta z_1 \tag{40}$$

$$S_b(T) = \sum_{1,2} \frac{1}{2\pi} \exp\left(-\frac{z_1^2 + z_2^2}{2}\right) \cos\left(\frac{1}{4} f_2\ z_2^2 \omega_d T\ \Omega_T^2\right) \Delta z_1 \Delta z_2$$

Eq. (26) readily follows because the two sums in Eq. (40), are respectively the Riemann sums of the two following integrals

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz = 1$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left(-\frac{z_1^2 + z_2^2}{2}\right) \cos\left(\frac{1}{4} f_2\quad z_2^2 \omega_d T \Omega_T^2\right) dz_1 dz_2 = \frac{\cos\left(\frac{1}{2}\arctan\alpha(T)\right)}{\sqrt[4]{\alpha^2(T)+1}}$$

where we have defined the function

$$\alpha(T) = \frac{1}{2} f_2\ \omega_d T\ \Omega_T^2 = \frac{1}{2\ln 2} \frac{\varepsilon_d^2}{BFI_T^2} f_3\ \omega_d T.$$

8 APPENDIX B

$$f_1(k_d D) = 1 + k_d D\ \frac{1 - \tanh^2 k_d D}{\tanh k_d D}, \tag{41}$$

$$f_2(k_d D) = 2 - \nu^2 + 8\,(k_d D)^2 \frac{\cosh 2k_d D}{\sinh^2 2k_d D}$$

$$f_3(k_d D) = \frac{8 + \cosh 4k_d D - 2\tanh^2 k_d D}{8\sinh^4 k_d D} - \frac{\left(2\cosh^2 k_d D + \frac{1}{2}\nu\right)^2}{\sinh^2 2k_d D\left(\frac{k_d D}{\tanh k_d D} - \frac{1}{4}\nu^2\right)}, \tag{42}$$

$$\nu = 1 + \frac{2k_d D}{\sinh 2k_d D}. \tag{43}$$

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