Extreme Events in Nonlinear Random Seas

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In this paper, the occurrence of extreme events due to the four-wave resonance interaction in weakly nonlinear water waves is investigated. The starting point is the Zakharov equation, which governs the dynamics of the spectral components of the surface displacement. It is proven that the optimal spectral components giving an extreme crest are solutions of a well-defined constrained optimization problem. A new analytical expression for the probability of exceedance of the wave crest is then proposed for the prediction of freak wave events. [DOI: 10.1115/1.2151202]

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Introduction

Single waves that are extremely unlikely as judged by the Raleigh distribution are called freak waves. The freak event that occurred on January 1, 1995 under the Draupner platform in the North Sea [1] provides evidence that such waves can occur in the open ocean. During this freak event, an extreme crest with an amplitude of 18.5 m occurred. The maximal wave height of 25.6 m was much more than twice the significant wave height of about 10.8 m.

Such a focusing of wave energy in a small area of the ocean can be justified by means of two linear mechanisms: time-space focusing and current focusing. In particular, the first mechanism can be explained by means of the theory of quasi determinism of Boccotti [2–6]. Boccotti showed that, if in a Gaussian sea a very high wave height occurs at some point in space and time, with high probability a well-defined quasi-deterministic wave group generates the high crest. In particular, the initial configuration of the wave group is such that in earlier stage of evolution of the group itself, the shortest waves are in front of the long waves. As time evolves, the long waves propagate faster and will catch up on the shorter waves, producing a focusing of energy at some point in space and time. The second linear mechanism of current focusing [7] requires almost unidirectional waves entering a zone of variable currents. However, ocean waves have a natural directional spreading and, consequently, the focusing effect is attenuated.

A third mechanism that can be a cause of freak waves is related to the four-wave interaction in weakly nonlinear water waves [8–13]. The weakly nonlinear energy transfer among nonresonant and resonant quartets is governed by the deterministic Zakharov integral-differential equation [14,15]. The nonlinear Schrödinger (NLS) equation [8] (valid for narrowband spectra) and the enhanced NLS equation [16,17] derived by Dysthe (valid for broader spectral bandwidth and larger steepness) are particular cases of the Zakharov equation. Current research focuses on understanding how extreme crests occur in a nonlinear random sea. In this context, from the solution of the NLS equation, Osborne et al. [18] showed that a typical wave field consists of a stable background intermittently disturbed by unstable rogue waves. Andonowati and van Groesen [19] proposed the concept of the maximal temporal amplitude (MTA) to determine the shape of the highest wave and Fedele and Arena [20,21] derived the form of the extreme wave within the second-order Stokes theory.

In this paper, a new mathematical model for the prediction of a freak wave is proposed. According to the wave dynamics dictated by the Zakharov equation, sufficient conditions for the occurrence of an extreme crest in weakly nonlinear water waves are given. Then a new asymptotic solution for the probability of exceedance of the crest height is proposed for the case of unidirectional deep water waves. This solution takes into account both second- and third-order effects.

Thus, a new numerical paradigm for the mathematical modeling of freak waves based on the Zakharov equation and the theory of quasi determinism of Boccotti is presented.

The Zakharov Equation

Let us consider weakly nonlinear water waves over a finite depth d. The free surface $\eta(\mathbf{x},t)$ is given by

$$\eta(\mathbf{x},t) = \frac{1}{2\pi}\sum \sqrt{\frac{\omega_n}{2g}}|B_n(t)|\exp[i(\mathbf{k}_n\cdot\mathbf{x} - \omega_n t)] + \varphi_n(t) + \text{c.c.} \tag{1}$$

where $\varphi_n(t)$ are arbitrary time-varying phase angles and the spectral component $B_n(t)$ is defined as

$$B_n(t) = |B_n(t)|\exp[i\varphi_n(t)] \quad \forall\, n \tag{2}$$

and $\mathbf{x}=(x,y)$ is the horizontal spatial vector. The linearized wave frequency ω_n is related to the wave number $\mathbf{k}_n$ through the linear dispersion relation $\omega_n^2/\mathbf{g}=|\mathbf{k}_n|\tanh(|\mathbf{k}_n|d)$. If third-order nonlinear effects are considered, then the spectral components $B_n(t)$ of the wave envelope satisfy the following discrete version of the Zakharov equation [14]:

$$\frac{dB_n}{dt} = -i\sum_{pqr} T_{npqr}\delta_{n+p-q-r}B_p^*B_qB_r\exp(i\Delta_{npqr}t) \tag{3}$$

Here, B^* denotes the complex conjugate of B, $\Delta_{npqr}=\omega_n+\omega_p-\omega_q-\omega_r$ and the kernel $T_{npqr}=T(\mathbf{k}_n,\mathbf{k}_p,\mathbf{k}_q,\mathbf{k}_r)$ is a real function of $\mathbf{k}_n,\mathbf{k}_p,\mathbf{k}_q,\mathbf{k}_r$. The kernel T_{npqr} is obtained by symmetrization as described in Krasitskii [15]. The generalized Kronecker delta $\delta_{n+p-q-r}$ denotes that summation is taken over those subscripts satisfying $\mathbf{k}_n+\mathbf{k}_p-\mathbf{k}_q-\mathbf{k}_r=0$.

The conserved quantities of Eq. (3) are the discrete Hamiltonian $\mathbf{H}$

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$$\mathbf{H}(\{B_n\}) = \sum_n \omega_n B_n B_n^* + \tfrac{1}{2}\sum_{npqr} T_{npqr}\delta_{npqr} B_n^* B_p^* B_q B_r \exp(i\Delta_{npqr}t) \quad (4)$$

the wave action $\mathbf{A}$ and wave momentum $\mathbf{M}=(M_x,M_y)$

$$\mathbf{A}(\{B_n\}) = \sum_n B_n B_n^*, \quad \mathbf{M}(\{B_n\}) = \sum_n \mathrm{k}_n B_n B_n^* \quad (5)$$

Here, $(\{B_n\})$ indicates the set of the spectral components $B_n(t)$. If the harmonic components form resonant quartets, i.e., $\omega_n+\omega_p-\omega_q-\omega_r=0$, the Hamiltonian $\mathbf{H}$ reduces to

$$\mathbf{H}(\{B_n\}) = \sum_n \omega_n B_n B_n^*$$

In deep water, resonant quartets can only occur for three-dimensional waves. In the following, only nonresonant conditions are considered; that is, $\omega_n+\omega_p-\omega_q-\omega_r \neq 0$. Thus, the Hamiltonian $\mathbf{H}$ is given by Eq. (4).

Sufficient Conditions for the Occurrence of an Extreme Crest

The wave surface in Eq. (1) is given by the superimposition of harmonic components nonlinearly interacting among each other, according to the evolution equation (3). At some initial time $t=-t_0$, $B_n(t)$ is set equal to some initial condition as

$$B_n(t=-t_0) = \tilde{B}_n = \tilde{b}_n \exp(i\tilde{\varphi}_n) \quad \forall\, n \quad (6)$$

Here, $\tilde{b}_n$ and $\tilde{\varphi}_n$ are initial amplitudes and phases to be defined later. As time varies, a nonlinear energy transfer among the harmonic components occurs and according to Eqs. (4) and (5), the Hamiltonian $\mathbf{H}$, the wave action $\mathbf{A}$, and the wave momentum $\mathbf{M}$ are conserved, that is

$$\mathbf{H}(\{B_n\}) = \mathbf{H}(\{\tilde{B}_n\}), \quad \mathbf{A}(\{B_n\}) = \mathbf{A}(\{\tilde{B}_n\}), \quad \mathbf{M}(\{B_n\}) = \mathbf{M}(\{\tilde{B}_n\}) \quad (7)$$

Assume that at $(\mathbf{x}=\mathbf{0},t=0)$ all the harmonic components in Eq. (1) are in phase, i.e.,

$$\varphi_n(t=0) = 0 \quad \forall\, n \quad (8)$$

This condition ensures that the nonlinear surface displacement $\eta(\mathbf{x},t)$ admits a stationary point at $(\mathbf{x}=\mathbf{0},t=0)$, i.e., $\nabla\eta|_{\mathbf{x}=0,t=0}=0$ and $\partial\eta/\partial t|_{\mathbf{x}=0,t=0}=0$. In fact, from Eq. (1) the spatial gradient and the partial derivative with respect to t of $\eta(\mathbf{x},t)$ are given by

$$\nabla\eta|_{\mathbf{x}=0,t=0} = \frac{1}{2\pi} i \sum \mathbf{k}_n \sqrt{\frac{\omega_n}{2g}} B_n(0)\exp[i\varphi_n(0)] + \mathrm{c.c.}$$

$$\left.\frac{\partial\eta}{\partial t}\right|_{\mathbf{x}=0,t=0} = \frac{1}{2\pi}\sum \sqrt{\frac{\omega_n}{2g}}\left[\left.\frac{dB_n}{dt}\right|_{t=0} - i\omega_n B_n(0)\right]\exp i\varphi_n(0) + \mathrm{c.c.} \quad (9)$$

If condition in Eq. (8) is imposed, invoking the evolution equation (3), one may evaluate the time derivative

$$\left.\frac{dB_n}{dt}\right|_{t=0} = -i\sum_{pqr} T_{npqr}\delta_{npqr}|B_p(0)||B_q(0)||B_r(0)|$$

and from Eq. (9) both the spatial gradient $\nabla\eta$ and the time derivative $\partial\eta/\partial t$ vanish at $(\mathbf{x}=\mathbf{0},t=0)$. Thus $\eta(\mathbf{x},t)$ has a stationary point at $(\mathbf{x}=\mathbf{0},t=0)$. Note that if the nonlinear effects are neglected, then condition in Eq. (8) is necessary and sufficient for the existence of an absolute maximum at $(\mathbf{x}=\mathbf{0},t=0)$ (see [6]).

In the following, sufficient conditions will be given such that the wave surface $\eta(\mathbf{x},t)$ attains its absolute maximum $H_{\max}$ at the stationary point $(\mathbf{x}=\mathbf{0},t=0)$; this maximum is the highest crest amplitude that can be attained during the wave evolution. From Eq. (1), the wave surface amplitude at any time t at $\mathbf{x}=\mathbf{0}$ is given by

$$\eta(\mathbf{0},t) = \frac{1}{\pi}\sum \sqrt{\frac{\omega_n}{2g}}|B_n(t)|\cos[\omega_n t + \varphi_n(t)]$$

Here, $\eta(\mathbf{0},t)$ admits the following upper bound

$$\eta(\mathbf{0},t) \leqslant \frac{1}{\pi}\sum \sqrt{\frac{\omega_n}{2g}}|B_n(t)| \quad (10)$$

If condition in Eq. (8) is imposed, at time $t=0$ the surface amplitude $H_{\max}=\eta(\mathbf{0},0)$ is given by

$$H_{\max} = \frac{1}{\pi}\sum \sqrt{\frac{\omega_n}{2g}}|B_n(0)| \quad (11)$$

If the amplitude $H_{\max}$ in Eq. (11) is forced to be greater than the right-hand side of the inequality (10), that is

$$\frac{1}{\pi}\sum \sqrt{\frac{\omega_n}{2g}}|B_n(0)| > \frac{1}{\pi}\sum \sqrt{\frac{\omega_n}{2g}}|B_n(t)|, \quad \forall\, t \quad (12)$$

then condition in Eq. (8) is sufficient for the existence of an absolute maximum at $(\mathbf{x}=\mathbf{0},t=0)$.

Now, define the dimensionless frequency $w_n=\omega_n/\omega_d$ and the dimensionless variables

$$X_n = \frac{|B_n(0)|}{H}\sqrt{\frac{\omega_d}{2g}}, \quad \tilde{X}_n = \frac{\tilde{B}_n}{H}\sqrt{\frac{\omega_d}{2g}}$$

where H is a characteristic wave amplitude (hereafter the highest linear crest amplitude) and ω_d a characteristic frequency. Then, the inequality in Eq. (12) is satisfied if one can determine a set of dimensionless harmonic amplitudes X_n satisfying the following optimization problem:

$$\max_{X_n\in\Re^N} \frac{1}{\pi}\sum \sqrt{w_n}X_n \quad X_n \geqslant 0 \quad (13)$$

subject to the constraints given in Eq. (7), which in terms of the X_n variables are expressed as

$$\sum_n w_n X_n^2 + \tfrac{1}{2}(\varepsilon_d\xi)^2 \sum_{npqr}\tilde{T}_{npqr}X_nX_pX_qX_r = \sum_n w_n\tilde{X}_n^2 + \tfrac{1}{2}(\varepsilon_d\xi)^2\sum_{npqr}\tilde{T}_{npqr}\tilde{X}_n\tilde{X}_p\tilde{X}_q\tilde{X}_r \exp -(i\tilde{\Delta}_{npqr}t_0 + i\tilde{\Phi}_{npqr}) \quad (14)$$

and

$$\sum_n X_n^2 = \sum_n \tilde{X}_n^2, \quad \sum_n \mathbf{k}_n X_n^2 = \sum_n \mathbf{k}_n\tilde{X}_n^2 \quad (15)$$

Here, $\tilde{\Phi}_{npqr}=\tilde{\varphi}_n+\tilde{\varphi}_p-\tilde{\varphi}_q-\tilde{\varphi}_r$, $\tilde{\Delta}_{npqr}=w_n+w_p-w_q-w_r$, $\varepsilon_d=|\mathbf{k}_d|\sigma$ is a characteristic wave steepness with $|\mathbf{k}_d|$ the wave number corresponding to the characteristic frequency ω_d, $\xi=H/\sigma$ is dimensionless amplitude, σ is the standard deviation of the wave surface and $\tilde{T}_{npqr}=T_{npqr}\delta_{n+p-q-r}/|\mathbf{k}_d|^3$. For a given choice of the initial time $t=-t_0$ and initial conditions $\{\tilde{X}_n\}$ and $\{\tilde{\varphi}_n\}$, one can solve the optimization problem (13) and determine the optimal spectral components $\{X_n\}$ along with the correspondent maximal wave amplitude $H_{\max}(t_0)$ given by (normalized with respect to the wave height amplitude H)

$$\frac{H_{\max}(t_0)}{H} = \frac{1}{\pi}\sum \sqrt{w_n}X_n \quad (16)$$

In simple words, starting from the time $t=-t_0$, the initial wave—with spectral components $\{\tilde{X}_n\}$ and phase angles

$\{\tilde{\varphi}_n\}$—nonlinearly evolves in space and time and at $(\mathbf{x}=\mathbf{0}, t=0)$ the highest wave crest amplitude $H_{\max}(t_0)$ is attained.

The condition (8) is also necessary for the occurrence of an extreme crest in weakly nonlinear waves provided the uniqueness of the solution of the constrained optimization problem (13), but this needs a formal proof, which will be not discussed here.

In the Euclidean space $\mathfrak{R}^N$ with N the number of harmonic components, the constraint in Eq. (14) represents a quartic hypersurface and the two constraints in Eq. (15) are both quadratic hypersurfaces. Their intersection manifold $\Gamma \in \mathfrak{R}^{N-4}$ is bounded since one of the hypersurfaces is a hypersphere. Because the objective function in Eq. (13) is linear in the X_n variables, then the point solution $P \in \mathfrak{R}^N$ of the optimization problem lies on the intersection manifold Γ. The solution of the optimization problem in Eq. (13) can be computed efficiently by means of the outer approximation method. This is a cutting-plane optimization technique that has been used successfully by Karatzas and Pinder [22] in groundwater quality management. The numerical implementation of this method is not straightforward; therefore, for the sake of simplicity, the MATLAB optimization toolbox has been used. The solution of Eq. (13) by the cutting-plane technique is an ongoing research, effort, which will be discussed in a future paper.

Nonlinear Crest Amplitude and Its Probability of Exceedance

Theory of Quasi Determinism. The theory of quasi determinism for the mechanics of linear wave groups was derived by Boccotti in the 1980s, with two formulations. The first one [2,6] enables us to predict what happens when a very high crest occurs in a fixed time and location (see also [23,24]); the second one [3,5,6] gives the mechanics of the wave group when a very large crest-to-trough height occurs.

The theory was then verified in the 1990s with some small-scale field experiments [4], both for waves in an undisturbed field and for waves interacting with structures. An alternative approach for the derivation of the quasi-determinism theory and a field verification off the Atlantic coast of the USA were proposed by Phillips et al. [25,26]. The first formulation of the theory (derived only for the time domain) was also reconsidered by Tromans et al. [27] and renamed as "New Wave theory."

Based on the first formulation of the theory, Boccotti showed that, if in a Gaussian sea state it is known that a very high local maximum occurs in some location and time, this implies with high probability that a well-defined wave group generates the high local maximum. In detail, if a local wave maximum of given elevation H occurs at a time $t=0$ at a fixed point $\mathbf{x}=\mathbf{0}$, with probability approaching 1, the surface displacement $\eta(\mathbf{x},t)$ tends asymptotically to the deterministic form

$$\eta_{\det}(\mathbf{x},t) = \frac{H}{\sigma^2}\int E(\mathbf{k},t)\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega(\mathbf{k})t)] \quad d\mathbf{k} + \text{c.c.} \quad (17)$$

as $H/\sigma\rightarrow\infty$, i.e., when the crest is very high with respect to the mean wave height. Here, $E(\mathbf{k})$ is the wave spectrum and $\sigma^2 = \int E(\mathbf{k})d\mathbf{k}$ is the variance of the Gaussian sea. An exceptionally high local maximum, with a very high degree of probability, is also a wave crest of its wave, since $\eta_{\det}(\mathbf{x},t)$ attains its absolute maximum at $(\mathbf{x}=\mathbf{0}, t=0)$. A direct consequence is that the number of wave crests exceeding a fixed threshold b tends to coincide with the number of local wave maxima exceeding it, provided the fixed threshold is very high; which, in turn implies the number of wave crests exceeding a very high threshold b tends to coincide with the number of b up-crossings (b_+); that is,

$$\frac{N_{cr}(b;T)}{N_+(b;T)} \rightarrow 1 \quad \text{as } b/\sigma \rightarrow \infty$$

Here, $N_{cr}(b;T)$ and $N_+(b;T)$ denote respectively the number of wave crests exceeding the threshold b and the number of b_+ in the very large time interval T. It is well known that [6]

$$N_+(b;T) \propto T\exp\left(-\frac{b^2}{2\sigma^2}\right)$$

then the probability of exceedance of a wave crest height admits the following asymptotic Rayleigh distribution (see also [28])

$$\Pr H > b = \frac{N_+(b;T)}{N_+(0;T)} = \exp\left(-\frac{b^2}{2\sigma^2}\right) \quad \frac{b}{\sigma} \rightarrow \infty \qquad (18)$$

For discrete spectra

$$E(\mathbf{k})d\mathbf{k} = \sum \frac{a_n^2}{2}\delta(\mathbf{k}-\mathbf{k}_n)d\mathbf{k} \qquad (19)$$

the deterministic group (17) admits the following expression:

$$\eta_{\det}(\mathbf{x},t) = \frac{H}{\sigma^2}\sum_n \frac{a_n^2}{2}\exp[i(\mathbf{k}_n\cdot\mathbf{x}-\omega_n t)] + \text{c.c.} \qquad (20)$$

This means that, in Gaussian seas with the wave spectrum defined such as in Eq. (19), if a high crest of height H occurs at $(\mathbf{x}=\mathbf{0}, t=0)$, with probability approaching one, the surface displacement $\eta_L(\mathbf{x},t)$ tends to assume the deterministic wave form $\eta_{\det}(\mathbf{x},t)$ as in Eq. (17). In the limit of $H/\sigma\rightarrow\infty$ the statistics of the wave crest height follows asymptotically the Rayleigh distribution (18).

Probability of Exceedance of an Extreme Crest. In the following unidirectional waves are considered, but the theory can be easily extended to the general case of three dimensional waves. At time $t=-t_0$ the initial conditions are set such that the nonlinear wave surface $\eta(\mathbf{x},t)$ is equal to the linear wave group $\eta_{\det}(\mathbf{x},t)$ in Eq. (20). Consequently, the initial spectral components and phase angles in Eq. (6) are set equal to

$$\tilde{B}_n = \pi\frac{H}{\sigma^2}\sqrt{\frac{2g}{\omega_n}}a_n^2, \quad \tilde{\varphi}_n = 0$$

Thus, the linear wave group (20), which in the absence of nonlinearities gives the highest crest amplitude H at $(\mathbf{x}=\mathbf{0}, t=0)$, nonlinearly evolves according to Eq. (3) and produces a different crest amplitude $H_{\max}$ at $(\mathbf{x}=\mathbf{0}, t=0)$. The solution of the optimization problem in Eq. (13) depends on the initial time $t=-t_0$, the set $\{\tilde{X}_n\}=\{\pi(\tilde{a}_n/\sigma)^2/\sqrt{w_n}\}$ of initial conditions, which determines the initial shape of the spectrum and the linear wave steepness $\varepsilon_H = \varepsilon_d\xi$ with $\xi=H/\sigma$ (see Eq. (14)). Here, ε_H is the steepness of the wave whose crest has the highest linear amplitude H according to Boccotti's theory (see Eq. (20)). Then from Eq. (16) the relation between the highest nonlinear crest amplitude $H_{\max}$ and the linear crest amplitude H is given by

$$\frac{H_{\max}}{\sigma} = 1 + \lambda[(\varepsilon_H, t_0, \{\tilde{X}_n\})]\xi \quad \xi\rightarrow\infty \qquad (21)$$

where the dimensionless parameter $\lambda(\varepsilon_H, t_0, \{\tilde{X}_n\})$ is defined by

$$\lambda(\varepsilon_H, t_0, \{\tilde{X}_n\}) = \frac{1}{\pi}\sum_n \sqrt{w_n}X_n - 1. \qquad (22)$$

Here, the set $\{X_n\}$, solution of the optimization problem (13), in the wave-number domain represents the Fourier spectrum of $\eta(\mathbf{x},t)$ at time $t=0$ when the highest crest occurs at $\mathbf{x}=\mathbf{0}$. At the initial time $t=-t_0$ the spectrum is given by the set $\{\tilde{X}_n\}$. Numerical investigations showed that for fixed values of the linear wave steepness ε_H, the parameter $\lambda(\varepsilon_H, t_0, \{\tilde{X}_n\})$ reaches a maximum $\lambda_{\max}$

$$\lambda_{\max} = \max_{t_0 \in \Re} \lambda(\varepsilon_H, t_0, \{\tilde{X}_n\}) \tag{23}$$

at approximately $\omega_d t_0 \propto \varepsilon_H^{-2}$, which is the time scale of the Benjamin-Feir instability [8,31]. In the limit of $\xi = H/\sigma \to \infty$, the statistics of the wave crest height follows asymptotically the Rayleigh distribution (18) and from Eqs. (21) and (23) the probability of exceedance of the nonlinear extreme wave crest $H_{\max}$ admits the following asymptotic expression:

$$\Pr(H_{\max} > h) = \exp\left(-\frac{h^2}{2\sigma^2(1+\lambda_{\max})^2}\right) \quad \frac{h}{\sigma} \to \infty \tag{24}$$

Note that $\lambda_{\max} > 0$ indicates self-focusing (i.e., the linear crest amplitude H increases due to third-order nonlinear interaction among free harmonics, i.e., harmonics satisfying the linear dispersion relation). As shown in the next section, for random seas with narrowband spectra, the parameter $\lambda_{\max}$ depends uniquely on an appropriate Benjamin-Feir index BFI (see [8]).

Third-order effects due to bound harmonics, i.e., harmonics which do not satisfy the linear dispersion relation, are neglected, but second-order effects due to bound harmonics are relevant. They break down the characteristic symmetry of Gaussian seas implying that higher crests are more probable than higher troughs [20,21,30]. If second-order effects due to bound harmonics are also taken into account, then Eq. (24) modifies as (see [29])

$$\Pr[H_{\max} > h] = \exp\left[-\frac{(1+\lambda_{\max})^2}{8\varepsilon_d^2\alpha^2}\left(1 - \sqrt{1 + \frac{4\varepsilon_d \alpha}{(1+\lambda_{\max})^2}\xi}\right)^2\right] \tag{25}$$

where one can assume $\alpha \approx 1/2$ on deep water [20,21].

Applications

Consider the case of unidirectional waves in deep water with a narrow band spectrum of dimensionless bandwidth ΔK. The harmonic components have dimensionless wavelength k separated by Δk, i.e., $k_n = 1 + n\Delta k$ with $n \in [-n_s, n_s]$ where n_s is the number of sideband components. Jansen [8] defines the Benjamin-Feir index (BFI)

$$\mathrm{BFI} = \frac{2\sqrt{2}\varepsilon_d}{\Delta K} \tag{26}$$

to characterize the nonlinear behavior of the random field. He showed that the intensity of the nonlinear interaction smoothly increases as the BFI increases without any bifurcation. As the frequency components of an initial wave packet change in time, energy flows from the central mode to the sideband modes. The energy eventually returns, restoring the wave to its initial state. This energy exchange occurs in time almost periodically, and it produces an effect of intermittence to the surface displacement: high crests occur intermittently in time, affecting the statistics of the wave crests, which tend to deviate from being Gaussian (see [32]). Extreme events become more probable due to the Fermi-Pasta Ulam recurrence, and the kurtosis of the wave distribution increases [32].

In the narrow-band limit the dimensionless frequency admits the following Taylor expansion for $\Delta K \ll 1$:

$$w_n = \sqrt{1 + n\Delta k} = 1 + \frac{n\Delta k}{2} - \frac{n\Delta k^2}{8} + o((n\Delta k)^2)$$

and the constraint in Eq. (14) relative to the Hamiltonian in Eq. (4) can be simplified as

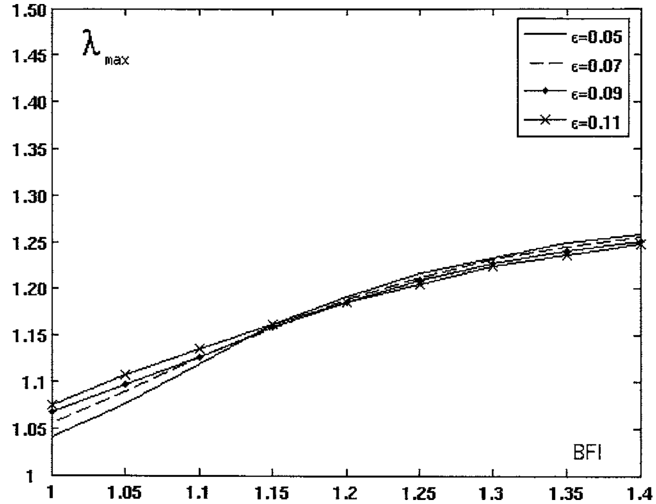

Fig. 1 The parameter $\lambda_{\max}$ as function of the Benjamin-Feir index BFI for different values of the wave steepness ε_d

$$-\Delta K^2 \sum_n (n/n_s)^2 X_n^2 + 4\varepsilon_H^2 \sum_{npqr} X_n X_p X_q X_r$$
$$= -\Delta K^2 \sum_n (n/n_s)^2 \tilde{X}_n^2 + 4\varepsilon_H^2 \sum_{npqr} \tilde{X}_n \tilde{X}_p \tilde{X}_q \tilde{X}_r \exp -(i\tilde{\Delta}_{npqr} t_0) \tag{27}$$

where the nonlinear transfer coefficient $\tilde{T}_{npqr}$ has been set equal to 1 [8]. By inspection of Eq. (26), one can easily observe that the solution set $\{X_n\}$ of the optimization problem (13) depends on the parameters $\Delta K, \varepsilon_H$ and the set $\{\tilde{X}_n\}$ of initial conditions (this set determines the initial shape of the spectrum at time $t = -t_0$). Moreover, the other constraints in Eq. (15) depend only directly on the initial set $\{\tilde{X}_n\}$. Thus, $\lambda_{\max}$ (see Eq. (23)) also depends on the parameters $\Delta K, \varepsilon_H$ and the set $\{\tilde{X}_n\}$. If one defines the Benjamin-Feir index BFI_H according to the wave steepness ε_H; that is,

$$\mathrm{BFI}_H = \frac{2\sqrt{2}\varepsilon_H}{\Delta K}$$

then both the parameter $\lambda_{\max}$ (see Eq. (23)) and the probability of exceedance in Eq. (24) rely uniquely on the BFI_H, as will be shown below.

Consider an initial wave spectrum with Gaussian shape

$$E(k) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(k-1)^2}{2\sigma^2}\right]$$

The dimensionless bandwidth of this spectrum is assumed to be equal to the relative width at the energy level of one half of the spectrum maximum, i.e., $\Delta K = 2\sigma\sqrt{2\ln 2}$. In the following, the effects to due second-order bound harmonics are neglected for the sake of simplicity. The continuous spectrum has been discretized in $N = 21$ spectral components (a central component and $n_s = 10$ sideband components) with an effective bandwidth equal to $2\Delta K = 10\sigma$. The optimization problem (13) is solved by using the MATLAB optimization toolbox.

For larger values of harmonic components $N > 21$ further studies are needed in order to formulate a robust numerical method for the solution of optimization problem (13), such as the outer approximation technique [23], but this will not be discussed here. In Fig. 1, the computed coefficient $\lambda_{\max}$ is plotted as a function of the parameter BFI_H for different values of the wave steepness ε_H. As one can see, $\lambda_{\max}$ depends uniquely on the Benjamin-Feir index BFI_H in the limit of small steepness.

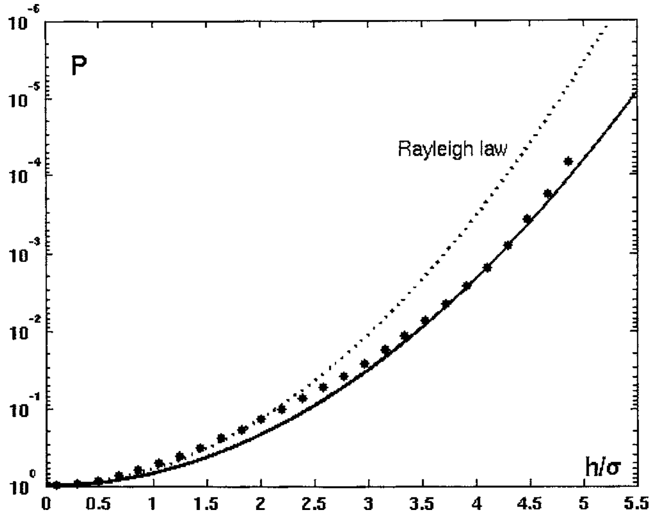

Fig. 2 Comparison between the analytical probability of exceedance of the wave crest Eq. (24) (solid line) and the empirical distribution computed from Monte Carlo simulations (dotted line). Here, BFI=1.15 and ε_d=0.05.

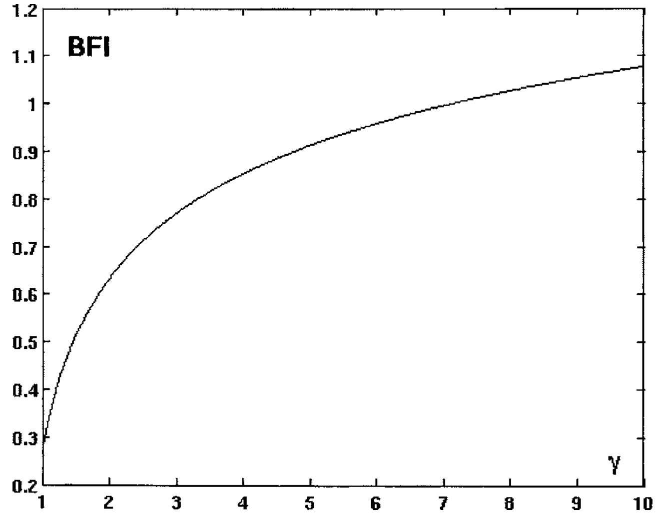

Fig. 3 The Benjamin-Feir index BFI as a function of the parameter γ of the JONSWAP spectrum

Monte Carlo simulations of the Zakharov equation (3) are performed to validate the analytical results for the case of $\mathrm{BFI}_H = 1.15$ and $\varepsilon_H = 0.05$. The Zakharov equation has been solved numerically by the Runge-Kutta method, and 2000 realizations of the nonlinear random field have been computed; each realization has duration of $\omega_d T = 8\varepsilon_H^{-2}$ so that the modulation instability occurred. The empirical crest distributions agree well with the analytical distribution (24) as one can see from the plots in Fig. 2.

It is obvious that these results are valid within the discrete Fourier approximation of the surface displacement with $N=21$ components. In reality, water waves have a continuous spectrum and therefore a larger number of harmonic components is needed in order to produce realistic results. As discussed above, a more robust optimization technique is needed in order to solve for a larger number of harmonic components.

JONSWAP Spectra

In this context the unidirectional waves, the Joint North Sea Wave Project (JONSWAP) spectrum [33], is adopted in the following form:

$$E(k) = \alpha k^{-3} \exp\left(-\frac{3}{2}k^2\right)\exp\left[\ln \gamma \exp\left(-\frac{(\sqrt{k}-1)^2}{2\chi^2}\right)\right]$$

Here, α is the Phillips parameter, γ is the enhancement coefficient, and for typical wind waves one can assume $\chi=0.08$. For $\gamma=1$ and $\alpha=0.0081$, the Pierson-Moskowitz spectrum is recovered. By Taylor expanding the spectrum around its peak, one obtains the following spectrum [34]:

$$E(k) = \frac{H_s^2}{16\pi}\frac{1}{1+(k-1)^2/\Delta K^2} \quad \Delta K = \sqrt{\frac{8\chi^2}{24\chi^2 + \ln \gamma}}$$

Here, ΔK is exactly the half width at half maximum of the JONSWAP spectrum (dimensionless bandwidth) and H_s is the significant wave height. The BFI parameter for the JONSWAP spectrum is plotted in Fig. 3 as a function of the enhancement coefficient γ. As γ increases, the spectrum becomes higher and narrower around the spectral peak and BFI increases.

By solving the optimization problem (13) one finds that the initial spectrum broadens symmetrically when the highest crest is formed. Due to the Benjamin-Feir instability the sideband harmonic components grow at the expense of the central components as one can see from Fig. 4 for the case of BFI=1.15 and $\varepsilon_H = 0.05$. If one relaxes the hypothesis of narrowband spectra, then a downshift of the spectrum also occurs [8].

Comparison to the Draupner Data

Consider now the data of the wave elevation measured at the Draupner field in the central North Sea during the storms in the period from December 31, 1994 to January 20, 1995. Joint frequency tables of successive wave crest heights and wave trough depths of the Draupner time series are provided by Wist et al. [1], and the empirical distributions are readily obtained. The draupner time series has significant wave height between 6.0 and 8.0 m, peak frequency $\omega_d=0.55$ rad/s, and mean wave period $T_m=9.1$ s. Consider a JONSWAP spectrum with $\gamma \approx 7$ for which BFI ≈ 1.05 and $\varepsilon_H \approx 0.04$. This choice gives the best fit with the experimental data as one can see from Fig. 5, where the comparison among the Draupner data, the third-order distribution in Eq. (25), and the second-order distribution (Eq. (25) with $\lambda_{max}=0$) is shown. Note that the analytical distribution tends to deviate from the second-order distribution, indicating an increasing in the kurtosis due to intermittency. But this effect is not strong since BFI ≈ 1.05. Moreover, the Draupner data are time series containing

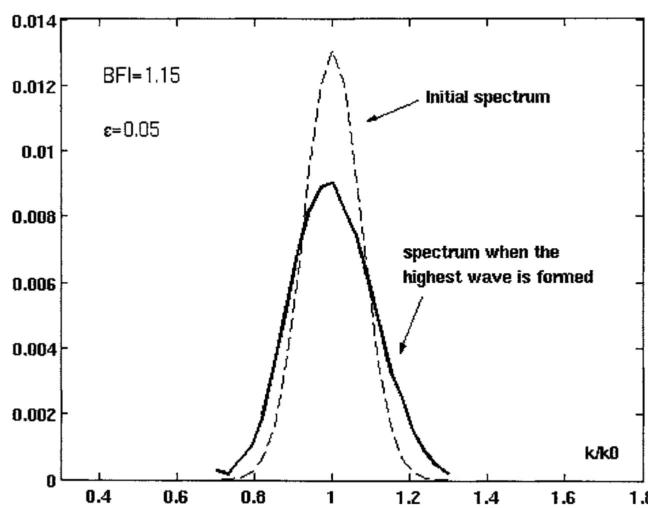

Fig. 4 Spectrum at the initial stage and when the highest crest is formed (BFI=1.15 and ε_d=0.05)

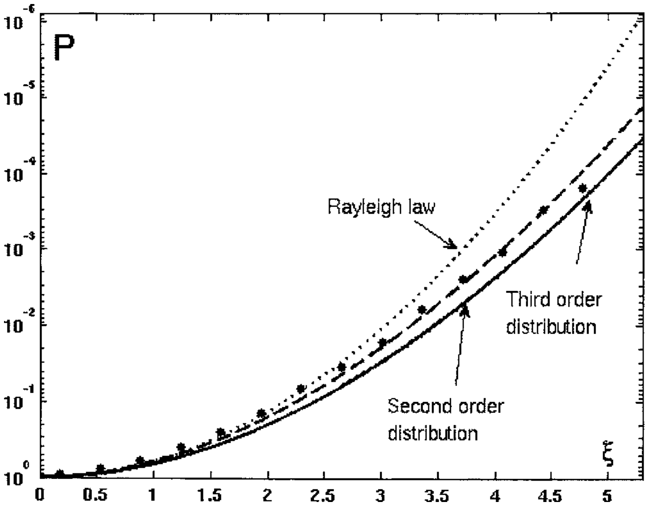

Fig. 5 Probabilities of exceedance

only one freak event. Thus, the tail of the empirical probability distribution deviates from the second-order analytical curve most likely because of statistical confidence error and not for intermittency effects: one freak wave event is not enough to compute the real tail of the probability of exceeding.

Conclusions

According to the Zakharov equation governing the dynamics the wave envelope of the surface displacement, the optimal spectral components that give an extreme crest are derived. They are solutions of a well-defined constrained optimization problem. By means of the theory of quasi determinism of Boccotti, the probability of exceedance of the wave crest is then obtained. Numerical probability distributions obtained by Monte Carlo simulations of the Zakharov equation are in agreement with the new analytical distribution for the case of narrowband spectra.

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