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# THE EQUIVALENT POWER STORM MODEL FOR LONG-TERM PREDICTIONS OF EXTREME WAVE EVENTS

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## ABSTRACT
We present a generalization of the equivalent triangular storm model of Boccotti for the long-term statistics of extreme wave events, where the actual storm is modeled in time $t$ by a power law ~ $|t|^{\lambda}$, with $\lambda$ as a shape parameter. Given the largest wave height $H_{max}$, we first identify the most probable storm in which the largest wave occurs. Then, we derive an explicit expression for the return period of a storm in which the maximum wave height exceeds a given threshold. We apply the new model for the analysis of wave measurements retrieved from some of the NOAA buoys in the Atlantic and Pacific oceans. We find that extreme event predictions based on the new model are less conservative than those from the peaks-over-thresholds method. Estimates of the expected maximum wave energy are also provided.

## INTRODUCTION
Stochastic modeling of significant wave height ($H_s$) time series provides the basis for efficient statistical methods for the prediction of extreme wave events during sea storms. In these type of analyses, the effects of the sea state on the short-term scales $T_s$~3 hours are nonlinearly cumulated to predict the wave conditions on the long-term time scales $T_l$~years. To do so, it is reasonable to assume that on the short-term scale $T_s$, the sea state is a homogenous and stationary stochastic field whose properties are fully characterized by the associated directional spectrum S(**k**) in the wave number space **k** and its spectral moments $m_j = \int S(\mathrm{k})\,\omega^j\,d\mathrm{k}$. Wave parameters such as $H_s$ and mean periods can be easily estimated from the sea surface time series or the associated wave spectrum. On the long-term scale $T_l$, we then have a succession of sea storms, where each storm, according to Boccotti (2000), is identified as a non-stationary sequence of sea states in which $H_s$ exceeds the fixed threshold $1.5\overline{H_s}$ and it does not fall below it for a continuous time interval greater than 12 hours. Here, $\overline{H_s}$ is the mean annual significant wave height for the examined site.

Given a succession of storm events in time, Boccotti (1986,2000) proposed the equivalent triangular storm (ETS) model to predict the return period of individual extreme wave events during sea storms. In the ETS model, a triangle of height $a$ and base $b$ is associated to each actual sea storm. The equivalence is realized by imposing that $a$ equals the maximum $H_s$ of the actual storm, and the associated $b$ is chosen such that the maximum expected wave height during the actual storm (Borgman, 1970, 1973) equals that of the ETS. In Boccotti's model, the long-term statistics of extreme events is uniquely characterized by three key elements: the Weibull distribution for the significant wave height $H_s$, the conditional average base $\bar{b}(a)$, and the $p_A(a)$ distribution of the ETS peak amplitudes $a$.

The ETS model led Boccotti (2000) to obtain analytical solutions for the return period $R(H_s>h)$ of a sea storm in which the maximum significant wave height $H_s$ exceeds $h$. Further, Boccotti (1989, 2000) derived the return period $R(H)$ of a sea storm in which the maximum individual wave height exceeds a fixed threshold $H$. Recently Arena (2001) extended this solution to deal with the return period $R(C)$ of a sea storm in which the maximum nonlinear crest height exceeds the fixed threshold $C$. The solutions for $R(H_s>h)$, $R(H)$ and $R(C)$ solve the problem of the prediction of extreme individual waves during sea storms. Arena & Pavone (2007) showed that the predictions from the ETS model are less conservative than those from the standard peaks-over-thresholds (POT) method. This is due to the property of the ETS model to accurately represent $H_s$ locally at storm peaks. Is it thus possible to improve the predictions of extremes by a more accurate modelling of storm peaks? Or equivalently, can we measure

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the goodness of the ETS model in predicting extremes in sea storms?

In this paper, we provide answers to these queries by presenting a generalization of the ETS model of Boccotti where each actual storm is modeled in time $t$ by a power law ~ $|t|^{\lambda}$, with $\lambda$ as a shape parameter. The paper is structured as follows. We first provide an overview of sea storm modeling, and then introduce the Equivalent Power Storm (EPS) model for the stochastic characterization of extreme wave events in sea storms. In particular, we derive an explicit expression for the return period of a storm in which the maximum wave height exceeds a given threshold. We then present results from the analysis of wave data measurements retrieved from some NOAA buoys moored off the Georgia coast, USA. To assess the quality of the EPS predictions, a sensitivity analysis is performed with respect to the shape parameter $\lambda$. We find a systematic pattern in the variations of the predictions as $\lambda$ is varied. To further validate our results, we compared the best EPS predictions for the return period $R(H_s > h)$ against those from the peaks-over-thresholds method (Goda, 1999, Van Vleddler et al. (1993). EPS estimates of the expected maximum wave energy are finally provided.

## SEA STORM MODELLING AT A GIVEN SITE IN TIME

To model sea storms, consider a time interval $\tau$ during which $N(\tau)$ storm events occur at the examined site in time. We assume that the probability that recorded $H_s$ stays above the threshold $h$ is given by the Weibull law

$$P(H_s > h) = \exp\left[-\left(\frac{h-h_l}{w}\right)^{u}\right]$$

which is defined for $h \geq h_l$. The parameters $h_l$, $w$ and $u$ are estimated by the iterative procedure of Goda (1999). We have considered the data of the buoys 46059 and 46006 moored far from the coast and the data of the buoys 46022, 46014, 46042, 46023, 46054 moored close to the coast as well (see Figure 1). Figure 2 shows the $P(H_s > h)$ of two buoys plotted on a Weibull paper. The parameters ($u$, $w$, $h_l$) of the $P(H_s > h)$ have been estimated for each buoy and are showed in Table 1.

The statistical properties of waves in the sea storm can be easily derived according to Borgman's theory (1973). Indeed, the probability of exceedance of the maximum wave height $H_{\max}$ in a sea storm is given by

$$P(H_{\max} > H) = 1 - \exp\left\{\int_0^D \frac{1}{\overline{T}[h(t)]} \ln\left[1 - P\left(H \mid H_s = h(t)\right)\right] dt\right\},$$

where $D$ is the storm duration, $h(t)$ is the significant wave height at the time $t$, $\overline{T}(h)$ is the mean period (Rice, 1944, 1945) and $P(H \mid H_s = h)$ is the probability of exceedance of the individual wave height given a a sea state with $H_s = h$. This is the Boccotti wave height distribution valid for finite band spectra (Boccotti 1981, 1997, 2000) that tends to the Rayleigh law for narrow band waves (Longuet-Higgins, 1952). The maximum expected wave height $\overline{H_{\max}}$ during the sea storm is given by

$$\overline{H_{\max}} = \int_0^{\infty} P(H_{\max} > H)\, dH\,.$$

**The Equivalent Power Storm (EPS) model**

We define an equivalent model whose significant wave height $h$ is defined by the power law

$$h = a\left[1 - \left(\frac{2|t|}{b}\right)^{\lambda}\right],$$

where $b$ is the duration of the storm, $a$ is the peak amplitude and $0 < \lambda < \infty$ is a shape parameter. The EPS model has one degree of freedom in $\lambda$ to better represent the actual storm peak. In particular, (4) has smooth peaks for $0 < \lambda < 1$, and sharp cusps for $\lambda \geq 1$. Further, as $\lambda$ increases from zero, peaks become smoothly sharper, and at $\lambda$=1 the ETS model of Boccotti with linear cusps is recovered. As $\lambda$ increases from 1, cusps become nonlinearly sharper. For the EPS model (4), the probability of exceedance of the maximum individual wave height follows from (2) as

(1)

$$P(H_{\max} > H; a, b) =$$

$$1 - \exp\left\{\frac{b}{\lambda a}\int_0^a \frac{\ln\left[1 - P\left(H / H_s = h\right)\right]}{\overline{T}(h)}\left(1 - \frac{h}{a}\right)^{-1+\frac{1}{\lambda}} dh\right\}.$$

Then, given an equivalent storm (4) with parameters $a$ and $b$, the maximum expected wave height $\overline{H_{\max}}$ follows by integration of (5), with respect to $H$, as in (3). We also introduce the joint probability density function (pdf) $p_{A,B}(a,b) = p_A(a)\, p_{B|A}(b,a)$ of $a$ and $b$ and define $p_{A,B}(a,b)\, db\, da$ as the fraction of equivalent storms having a duration in $[b, b+db]$ and peak amplitude in $[a, a+da]$ during a long time interval $\tau$. Note that the conditional average duration

$$\overline{b}(a) = \int_0^{\infty} p_{B|A}(b,a)\, db$$

(2)

and the peak distribution $p_A(a)$ are the key parameters that uniquely characterize the EPS model.


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The estimation of $a$, $b$, $\overline{b}(a)$ and $p_A(a)$ proceed from the storm data by imposing the following equivalences between the EPS model and the actual storm. The height $a$ (storm intensity) of the EPS is equal to the maximum significant wave height during the actual storm. The duration $b$ is chosen so that $\overline{H_{\max}}$ of the EPS equals the maximum expected wave height of the actual storm. Then, $\overline{b}(a)$ can be estimated from the $N(\tau)$ pairs $(a_j, b_j)$. Figure 3 shows the actual storm, the associated EPS and the respective exceedance probabilities $P(H_{\max} > H)$ computed from (2) and (5), respectively. These two probabilities are practically the same, implying the full equivalence between each actual storm and the associated EPS. Note that $b$ is correlated with the storm peak $a$. In particular, short durations b~12-24 hours are typical of very strong and fast actual storms (see Figure 3*a-b* where two sea storms having a small base are shown). Long durations $b$~100-200 hours generally occurs in sea storms having a large single peak, or in a sea storm with multi peaks (see Figure 3*c*).

Finally, the pdf $p_A(a)$ is obtained by imposing that the total time during which the significant wave height is, during a given interval $\tau$, above any given threshold $h$ is equal in both the actual storm and its equivalent EPS model (see Fig. 2). For the actual storm, the time $T_R$ during which $H_s$ stays above $h$ is given by

$$T_R(h) = \tau \Pr\left(H_s > h\right). \tag{7}$$

For the EPS model, the time $T_{GES}$ during which $H_s$ is above $h$ can be derived as follows. Consider first

$$dN(a,b) = N(\tau) p_A(a) p_{B|A}(b,a) db da \tag{8}$$

as the number of equivalent storms having a duration in $[b, b+db]$ and peak amplitude in $[a, a+da]$, and recall that $N(\tau)$ is the total number of storms in $\tau$. From Fig. 2,

$$T_{GES}(h) = \int_{a=h}^{\infty} \int_{b=0}^{\infty} t_s(h,a,b)\, dN(a,b) , \tag{9}$$

where $t_s(h,a,b)$ is the time when $H_s$ stays above $h$, during an equivalent storm. This follows from (4) as (see also Fig. 1)

$$t_s(h,a,b) = b\left(1 - \frac{h}{a}\right)^{1/\lambda}. \tag{10}$$

Thus, from (6) and (8), (9) can be further simplified as

$$T_{GES}(h) = N(\tau) \int_h^{\infty} \overline{b}(a) \left(1 - \frac{h}{a}\right)^{1/\lambda} p_A(a)\, da . \tag{11}$$

To derive $p_A(a)$, we impose that $T_{GES}(h) = T_R(h)$. This leads to the following integral Volterra equation of first kind

$$\tau P(h) = N(\tau) \int_h^{\infty} \overline{b}(a) \left(1 - \frac{h}{a}\right)^{1/\lambda} p_A(a)\, da , \tag{12}$$

where $P(h) = \Pr(H_s > h)$. The analytical solution of (12) for $p_A(a)$ is given by (13) (see appendix)

$$p_A(a) = \frac{\tau}{N(\tau)} \frac{a}{\overline{b}(a)} G(\lambda, a) , \tag{13}$$

where

$$G(\lambda,a) = \begin{cases} \dfrac{\sin(\pi/\lambda)}{\pi/\lambda} \displaystyle\int_1^{\infty} \left.\frac{d^2 P}{dz^2}\right|_{ax} (x-1)^{-1/\lambda}\, dx, & if\ \ \lambda > 1 \\ \dfrac{d^2 P}{da^2} & if\ \ \lambda = 1 \\ \dfrac{(-1)^n a^n}{n!} \dfrac{\sin \pi\mu}{\pi\mu} \displaystyle\int_1^{\infty} \left.\frac{d^{n+2} P}{dz^{n+2}}\right|_{ax} (x-1)^{-\mu}\, dx, & if\ \ \lambda < 1, \end{cases} \tag{14}$$

and the first three derivatives of $P(z)$ are given by

$$\frac{dP}{dz} = -\frac{u}{w}\left(\frac{z}{w}\right)^{u-1} P, \qquad \frac{d^2 P}{dz^2} = \frac{dP}{dz}\frac{1}{z}\left[-u\left(\frac{z}{w}\right)^u + u - 1\right],$$

and

$$\frac{d^3 P}{dz^3} = \frac{d^2 P}{dz^2}\frac{u}{z}\left[2 - \frac{2}{u} - \left(\frac{z}{w}\right)^u + \frac{u-1}{1 - u + u\left(z/w\right)^u}\right].$$

Note that for the case $\lambda < 1$, the integer $n > 1$ and the real number $0 < \mu < 1$ are such that $1/\lambda = n + \mu$. If $\mu = 0$, from (14)

$$G(\lambda, a) = -\frac{(-1)^n a^n}{n!} \frac{d^{n+1} P}{da^{n+1}}, \quad if\ \ \lambda = 1/n < 1.$$

## The return period $R(H_s > h)$

Consider a time interval $\tau$ during which $N(\tau)$ storm events occur. According to the EPS model, $dN(a,b)$ of (8) is the number of equivalent storms having a duration in $[b, b+db]$ and peak amplitude in $[a, a+da]$. Thus, the number of storms $N(\tau; a > h)$ with the peak significant wave height $a$>$h$ is given by


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$$N(\tau;a>h)=\int_{a=h}^{\infty}\int_{b=0}^{\infty}dN(a,b)=N(\tau)\int_{h}^{\infty}p_A(a)\,da\;. \tag{15}$$

Since $a$ is the storm peak amplitude, we define the return period $R(H_s>h)$ of a storm in which the maximum significant wave height $H_s$ exceeds the threshold $h$ as

$$R(H_s>h)=\frac{\tau}{N(\tau;a>h)}=\frac{\tau}{N(\tau)\int_{h}^{\infty}p_A(a)\,da}. \tag{16}$$

The mean persistence of the significant wave height $H_s$ above the threshold $h$ in the storms where this threshold is exceeded, is given by the general expression (Boccotti, 2000):

$$\overline{D}(h)=R(H_s>h)P(H_s>h)\;. \tag{17}$$

This yields the alternative formulation $R(H_s>h)$ as

$$R(H_s>h)=\frac{\overline{D}(h)}{P(H_s>h)}\;. \tag{18}$$

Some examples of the mean persistence $\overline{D}(h)$ off the Californian coast are showed in Figure 8.

## 4. Extreme wave events in sea storms

We now proceed with the statistical properties of the largest wave occurring during storms. We first derive the conditional pdf $p_{A|Hmax}(H,a)$, which is interpreted as the fraction of equivalent storms whose largest wave height $H_{\max}$ is equal to $H$ and the peak intensity $A$ is in $[a,a+da]$. To do so, consider the number $dN_w(H,a,b)$ of equivalent storms, with a duration in $[b,b+db]$ and peak amplitude in $[a,a+da]$, during which the maximum wave occurs with an height $H_{max}$ in $[H,H+dH]$, that is

$$dN_w(H,a,b)=dN(a,b)\,p(H_{\max}=H;a,b)dH\;. \tag{19}$$

Here, $p(H_{\max}=H;a,b)$ is the pdf of $H_{max}$ that follows from (5) and $dN$ is given in (8). From (19), we define the conditional pdf $p_{A|Hmax}(a,H)$ as

$$p_{A|H_{\max}}(a,H)da=\frac{\int_{b=0}^{\infty}dN_w(H,a,b)}{\int_{a=0}^{\infty}\int_{b=0}^{\infty}dN_w(H,a,b)}= \;. \tag{20}$$

$$=\frac{p_A(a)\;p\left(H_{\max}=H;a,\overline{b}(a)\right)da}{\int_{0}^{\infty}p_A(a)\,p\left(H_{\max}=H;a,\overline{b}(a)\right)da}$$

(16)

The conditional expected storm intensity, given the largest wave height $H$, is denoted as $a_c(H)=\overline{A\,|\,H_{\max}}$ and the associated variance is $\sigma^2_{a,c}(H)=\mathrm{var}\left(A\,|\,H_{\max}\right)$. We shall show via comparisons with buoy data that

(17)

$$a_c(H)\approx\frac{H}{2},\qquad \frac{\sigma_{a,c}(H)}{a_c(H)}<<1,\quad for\;\; H>>1. \tag{21}$$

(18)

Thus, the most probable height of the largest wave in a storm is as twice as its peak amplitude. To obtain the exact analytical solution of $R(H_{\max}>H)$, consider the number $N_w(H)$ of storms in which the largest wave occurs with a height greater than $H$. This is given by

$$N_w(H)=\int_{a=0}^{\infty}\int_{b=0}^{\infty}dN_w(H,a,b). \tag{22}$$

Then, we can define by (23)

$$R[H_{\max}>H]=\frac{\tau}{N_w(H)}=\frac{\tau}{N(\tau)\int_{h}^{\infty}p_A(a)P\left(H_{\max}>H;a,\overline{b}(a)\right)da}\;.$$

The return periods $R(H_s>h)$ for some locations off the Californian coast are showed in Figure 7. Table 3 shows also the significant wave heights for fixed values of the return period $R(H_s>h)$, calculated for the buoys of Figure 1.

**3 Sea storm modelling over an area in time**

The EPS model can also be extended to predict extreme wave events over a given area. To do so, we just need to define the exceedance probabilities (2) and (5) of the actual and equivalent storms appropriate for the maximum wave height $H_{\max}$ over a given region $\Omega$. To do so, we model the wave height $H_{\max}(\Omega,t)$ as a non non-stationary process depending on a wave parameter $H_s(t)$, and define the wave height exceedance according to (24)

$$P[H_{\max}(\Omega)>H]=\int_{0}^{\infty}\Pr\{H_{\max}(\Omega)>H\,|\,H_s=w\}\,p(H_s=w)dw\;.$$


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Here, the probability $p(H_s = w)dw$ is interpreted as the fraction of time during which $H_s(t)$ stays between $w$ and $w+dw$, that is

$$p(H_s = w)dw = \lim_{\tau\to\infty} \frac{\text{time when } H_s \text{ is in } [w, w+dw] \text{ in } \tau}{\tau}, \quad (25)$$

and $\Pr\{H_{\max}(\Omega) > H \mid H_s = w\}$ is the probability that the wave height exceeds the threshold $H$ over the region $\Omega$ in a sea state η with $H_s$=$w$. This is given by (Adler 1981, Adler & Taylor 2007, Piterbarg 1995)

$$\Pr\{H_{\max}(\Omega) > H \mid H_s = w\} = A_\Omega \left(2\pi\right)^{-3/2} \sigma^{-2} \left|\Lambda\right|^{1/2} \xi \, e^{-\xi^2/2} \quad (26)$$

where, $\xi = H/\sigma$ is the normalized threshold amplitude, $\sigma$ is the standard deviation of η, $A_\Omega$ is the area of region $\Omega$, and $\Lambda$ is the covariance matrix of the gradient $\nabla\eta$. For each actual storm of duration $D$, during which $H_s$ varies according to $h(t)$, we can write

$$P\left[H_{\max}(\Omega) > H\right] = \int_0^D \Pr\{H_{\max}(\Omega) > H \mid H_s = h(t)\} p(H_s = h(t)) h'(t)\, dt \quad (27)$$

where prime denotes time derivative. For the EPS model,

$$P\left[H_{\max}(\Omega) > H\right] = \frac{2b}{\lambda a} \cdot \int_0^b \Pr\{H_{\max}(\Omega) > H \mid H_s = h\} \left(1 - \frac{h}{a}\right)^{-1+\frac{1}{\lambda}} p(H_s = h)\, dh. \quad (28)$$

## Extreme storms recorded off the Californian coast

The analysis of the sea storm recorded off the Californian coast has been performed. In particular, for each actual storm the height and the base of the equivalent triangular storm have been calculated. Some examples of ETS are showed in Figure 3. The severest storm is showed in Figure 3*a*: it was recorded by buoy 46006 and it had intensity of 16.3m and duration of 32.2hours.

Finally, the mean height $a_{10}$ and the mean base $b_{10}$ of the *N'* strongest ETS at the examined location are obtained for each buoy (*N'* is equal to 10 times the number of observation years). The values of $a_{10}$ and $b_{10}$ are showed in Table 1. As we can see, we find the highest values of $a_{10}$ (and therefore the strongest storm) for the buoys 46006 and 46059 far from the coast (see Figure 1). For the buoys close to the coast, $a_{10}$ is generally smaller, as we can see from Table 1. From Table 1 we can see also that $b_{10}$ (that is the mean duration of the strongest storm) is more uniform than $a_{10}$.

## Storm duration

The storm duration can be estimated by using the exponential base-height regression proposed by Boccotti (2000) (see also Arena and Barbaro, 1999):

$$\overline{b}(a) = K_1 b_{10} \exp\left(K_2 \frac{a}{a_{10}}\right)$$

where $K_1$ and $K_2$ are characteristic parameters of the location (these parameters were estimated in the Central Mediterranean Sea, in the Northwest Atlantic Ocean and in the Northeast and Central Pacific Ocean – see Boccotti, 2000 and Arena and Barbaro, 1999). Figure 4 shows the relation between the storm duration and the storm intensity for the NOAA buoy 46006. Each point in the figure represents an actual storm having intensity $a$ and duration $b$. The parameters $K_1$, $K_2$, $a_{10}$ and $b_{10}$ of $\overline{b}(a)$ for the examined locations off the Californian coast are showed in Table 1.

## Comparisons with the Total Sample Method

Consider the estimated $P(H_s > h)$ from $H_s$ data at the sampling rate, say $\Delta t_{\text{samp}}$. Given the large time interval $\tau$, $\tau P(H_s > h)/\Delta t_{\text{samp}}$ is an estimate of the number of records in which $H_s$ is greater than $h$, during $\tau$. In the Total Sample Method (TSM), the return period $R_{ts}(H_s > h)$ of a sea storm in which the maximum significant wave heights exceeds $h$ is computed as that of a sea state whose $H_s > h$, viz.

$$R_{ts}(H_s > h) = \frac{\Delta t_{\text{samp}}}{P(H_s > h)}.$$

From (18), we clearly see that $R_{ts}(H_s > h)$ is obtained assuming the mean persistence $\overline{D}(h) = \Delta t_{\text{samp}}$. For NOAA buoys data, $\Delta t_{samp}$=1 hour. In reality $H_s$ can stay above $h$ for several hours. Indeed, $\overline{D}(h)$ equals dozens of hours for average $h = 1.5\overline{H_s}$, and it decreases as $h$ increases, as was pointed out by Graham (1982), Sobey and Orloff (1999) and Boccotti (2000). This tendency can be clearly seen in Fig. 8. As a consequence, the TSM tends to overestimate the extreme significant wave height during severe storms, as shown in both Figure 7 and Table 3, where we report both the predictions based on the TSM and EPS models, respectively.

## CONCLUSIONS

In this paper a bi-parametric analysis of sea storms by using the data of some NOAA buoys moored off the California is done.


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The *equivalent Power storm* (EPS) model is presented. We have estimated the return period $R(H_s > h)$ of a sea storm in which the maximum wave height exceeds the fixed threshold $h$ (Boccotti, 2000), assuming the lower-bounded three parameters Weibull distribution for the significant wave height. Therefore we have compared the EPS model prediction to the total sample method predictions finding that the latter tends to overestimate the significant wave height for fixed return period.

Finally in order to predict the direction of the strongest sea storms, a directional analysis is done, using the data from the directional NOAA buoy 46042.

## Appendix

In (12) set

$$f(a) = \frac{N(\tau)\bar{b}(a)}{\tau\, a^{1/\lambda}}\, p_A(a)\,, \qquad \text{(a1)}$$

to obtain an integral Volterra equation of first kind for $f(a)$, that is

$$P(h) = \int_h^\infty f(a)(a-h)^{1/\lambda}\, da\,. \qquad \text{(a2)}$$

After we solve for $f(a)$, the solution for $p_A(a)$ easily follows from (a1). To solve for (a2) we distinguish three cases: a) $\lambda = 1$, b) $\lambda > 1$, and c) $0 < \lambda < 1$.

***Case a):*** $\lambda = 1$

Note that (a2) reduces to

$$P(h) = \int_h^\infty f(a)(a-h)\, da\,. \qquad \text{(a3)}$$

The solution of $f(a)$ proceed by differentiating both members of (a3) twice, with respect to $h$, and setting $h=a$. This yield

$$f(a) = \frac{d^2 P}{da^2}\,. \qquad \text{(a4)}$$

***Case b):*** $\lambda > 1$

Consider the ansatz

$$f(a) = \int_a^\infty g(z)\, H(z-a)\, dz, \qquad \text{(a5)}$$

where $g(z)$, for $z \ge 0$, and $H(z-a)$, for $z \ge a$, are arbitrary functions. Substituting (a5) into (a2) yields the new integral equation for $g$

$$P(h) = \int_{a=h}^\infty \int_{z=a}^\infty g(z)\, H(z-a)\,(a-h)^{1/\lambda}\, da\, dz. \qquad \text{(a6)}$$

The domain of integration of the double integral in (a6) is shown in Fig. A1. If, for given $z$, we first integrate along $a$ within the limits $[h,z]$, and proceed with the integration along $z$, between $h$ and $\infty$, then (a6) can be written as

$$P(h) = \int_h^\infty g(z)\, K(h,z)\, dz\,, \qquad \text{(a7)}$$

where the kernel $K$ is defined as

$$K(h,z) = \int_h^z H(z-a)\,(a-h)^{1/\lambda}\, da\,. \qquad \text{(a8)}$$

The integral equation (a7) can be easily solved for $g(z)$ if $H$ is chosen so that $K \sim (z-h)$. To do so, consider the change of variables

$$a = \frac{z+h}{2} + \frac{z-h}{2}\cos\theta \qquad \text{(a9)}$$

that transforms the kernel (a8) to (a10)

$$K(h,z) = -\int_\pi^0 H\left[\frac{z-h}{2}(1-\cos\theta)\right]\left(\frac{z-h}{2}\right)^{1/\lambda}(1+\cos\theta)^{1/\lambda}\,\frac{z-h}{2}\sin\theta\, d\theta$$

If we choose

$$H(z-a) = \frac{1}{(z-a)^{1/\lambda}}\,, \qquad \text{(a11)}$$

Then, for $\lambda > 1$,

$$K(h,z) = \frac{z-h}{2}\int_0^\pi \left(\frac{1+\cos\theta}{1-\cos\theta}\right)^{1/\lambda} \sin\theta\, d\theta = \frac{\pi(z-h)}{\lambda \sin(\pi/\lambda)}, \qquad \text{(a12)}$$

and (a7) simplifies to

$$P(h) = \frac{\pi/\lambda}{\sin(\pi/\lambda)}\int_h^\infty g(z)(z-h)\, dz\,. \qquad \text{(a13)}$$

This is the same type of Volterra equation as in case (a). We thus solve for $g(z)$ by differentiating both members of (a13) twice with respect to $h$, and then setting $h=z$. This yield

$$g(z) = \frac{\sin(\pi/\lambda)}{\pi/\lambda}\frac{d^2 P}{dz^2}\,. \qquad \text{(a14)}$$


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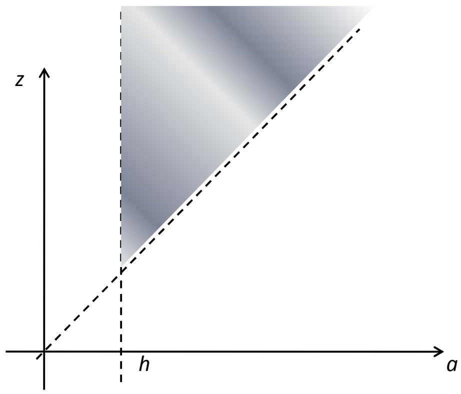


**Figure A1:** Domain of integration of the double integral in (a4)

From (a11) and (a14), (a5) yields the final solution of (a2)

$$f(a)=\frac{\sin(\pi/\lambda)}{\pi/\lambda}\int_a^{\infty}\frac{d^2P}{dz^2}\frac{1}{(z-a)^{1/\lambda}}dz, \qquad \lambda>1. \quad \text{(a15)}$$

***Case c):*** $0<\lambda<1$

Find an integer $n>1$ and a real number $0<\mu<1$ such that

$$\frac{1}{\lambda}=n+\mu\,; \quad \text{(a16)}$$

then (a2) becomes

$$P(h)=\int_h^{\infty}f(a)(a-h)^{n+\mu}\;da\,. \quad \text{(a17)}$$

We first differentiate *n* times both members of (a17) and get

$$\frac{(-1)^n}{n!}\frac{d^nP}{dh^n}=\int_h^{\infty}f(a)(a-h)^{\mu}\;da, \qquad 0<\mu<1\,. \quad \text{(a18)}$$

This is the same type of integral equation as in case b). Thus,

$$f(a)=\frac{(-1)^n}{n!}\frac{\sin(\pi\mu)}{\pi\mu}\int_a^{\infty}\frac{d^{n+2}P}{dz^{n+2}}\frac{1}{(z-a)^{\mu}}dz. \quad \text{(a19)}$$

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