31° Convegno Nazionale di Idraulica e Costruzioni Idrauliche
Perugia, 9-12 settembre 2008

EULER CHARACTERISTICS & MAXIMA OF OCEANIC SEA STATES

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ABSTRACT

We propose a novel Variational Wave Acquisition Stereo System (VWASS) that exploits new stereo reconstruction techniques for accurate estimates of the space-time dynamics of oceanic sea states. The rich information content of the acquired three-dimensional video data is used to make predictions on directional spectra and large waves over an area. To do so, we present a new statistical analysis based on the Euler Characteristic of excursion sets of random fields. The broader impact of these results for oceanic applications is finally discussed.

1 INTRODUCTION

The prediction of large waves is typically based on the statistical analysis of time series of the wave surface displacement retrieved from wave gauges, ultrasonic instruments or buoys at a fixed point *P* of the ocean. However, the largest wave crest predicted in time at *P* underestimates the highest crest expected over the area nearby *P*. Indeed, large waves travel on top of wave groups, and the probability that the group passes at its apex through *P* is practically null. The large crest height recorded in time at *P* is simply due to the dynamical effects of a group that focuses nearby that location forming a larger wave crest. Can we predict the largest wave expected over a given area?

In this paper, we address this question by proposing a novel variational Wave Acquisition Stereo System (VWASS) for the reconstruction of the water surface of oceanic sea states. The rich information content of the acquired three-dimensional video data is exploited to compute reliable estimates of the expected global maximum (largest crest height) over an area using the Euler Characteristic of random excursion sets (Adler 1981, Taylor & Adler 2007). The paper is structured as follows. We first discuss the mathematical formulation of VWASS and how it is used in applications. We then introduce Euler Characteristics (*EC*) of the excursion sets (Adler 1981)

$$A_{\eta,h} = \{(x,y) \in S : \eta(x,y) > h\} \tag{1}$$

of a two dimensional (2D) random field η, and its relation to the expected number of maxima and *h*-upcrossings over an area S . We then analyze the *EC* of the excursion sets of the spatial snapshots η of oceanic sea states acquired via VWASS. We present new estimates from video data of both directional wave spectra and empirical exceedance probabilities of the global maximum of η over *S*. The broader impact of these results to oceanic applications is finally discussed.

2 THE STEREO VARIATIONAL GEOMETRIC METHOD

The reconstruction of the wave surface from stereo pairs of ocean wave images is a classical problem in computer vision commonly known as the correspondence problem (Ma et al. 2004). Its solution is based on epipolar geometry techniques that find corresponding points in the two images, from which one obtains the estimate of the real point in the three dimensional terrestrial coordinate system. The Wave Acquisition Stereo System (WASS) developed by Benetazzo (2006) utilizes the 'epipolar algorithm' for the space-time reconstruction of the sea surface. However, this approach may fail to provide a smooth surface reconstruction because of "holes" corresponding to unmatched image regions (Ma et al. 2004, Benetazzo 2006). For example, this can occur when, at a given point on the water surface, the same amount of light is received from all possible directions and reflected towards the observer causing a visual blurring of the specularities of the water. This is typical of cloudy days, and the water surface is said to support a Lambertian radiance function (Ma et al. 2004). We address this problem by proposing a novel formulation of WASS based on variational principles (VWASS). Under the assumptions of a Lambertian surface, following the seminal work by (Faugeras et al. 1998), the 3-D reconstruction of the water surface is obtained in the context of active surfaces by evolving an initial surface through a PDE derived from the gradient descent flow of a cost functional designed for the stereo reconstruction problem.

To be more specific, the energy being *maximized* is the normalized cross correlation between the image intensities obtained by projecting the same water surface patch onto both image planes of the cameras. It is clear that such energy depends on the shape of the water surface. Therefore, the active surface establishes an evolving correspondence between the pixels in both images. Hence, the correspondence will be obtained by evolving a surface in 3-D instead of just performing image-to-image intensity comparisons without an explicit 3-D model of the target surface being reconstructed.

To infer the shape of the water surface $\eta(x, y)$ at the location (x,y) over an area *S*, we set up a cost functional on the discrepancy between the projection of the model surface and the image measurements. As previously announced, such cost is based on a cross correlation measure between image intensities, which will be noted as $E_{\text{data}}(\eta)$. We conjecture that, to have a well-posed problem, a regularization term that imposes a geometric prior must also be included, $E_{\text{geom}}(\eta)$. We consider the cost functional to be the (weighted) sum:

$$E(\eta) = E_{\text{data}}(\eta) + E_{\text{geom}}(\eta). \tag{2}$$

In particular, the geometric term favors surfaces of least area:

$$E_{\text{geom}}(\eta) = \int_\eta dA\,. \tag{3}$$

The data fidelity term may be expressed as

$$E_{data}(\eta) = \int_\eta \left(1 - \frac{\langle I_1, I_2 \rangle}{|I_1| \cdot |I_2|} \right) dA \,, \tag{4}$$

where η is the wave surface region within the field of views of both cameras, and $\langle I_1, I_2 \rangle$ is the cross-correlation between the image intensities I_1 and I_2 .

To find the surface η that minimizes *E*, we start from an initial estimate of the surface at time $t = 0$, η_0, and set up a gradient flow based on the first variation of *E* that will make the surface evolve towards a minimizer of *E*, hopefully converging to the desired water surface shape.

Based on the theorem in (Faugeras et al. 1998) that says that for a function $\Phi : R^3 \times R^3 \rightarrow R^+$ and the energy

$$E = \int_\eta \Phi(X, N) dA, \tag{5}$$

where *N* is the unit normal to η at *X*, the flow that minimizes *E* is given by the evolution PDE

$$\eta_t = \beta N \,, \tag{6}$$

where η_t is the derivative of η with respect to a fictitious time variable and the speed β in the normal direction to the surface that drives the evolution is

$$\beta = 2H(\Phi - \Phi_N \cdot N) - \Phi_X \cdot N - trace\left[(\Phi_{XN})_{T_\eta} + dN \circ (\Phi_{NN})_{T_\eta} \right] \tag{7}$$

All quantities are evaluated at the point $\eta = X$ with normal *N* to the surface. *H* denotes the mean curvature. Φ_X, Φ_N are the first-order derivatives of Φ , while Φ_{XN}, Φ_{NN} are the second-order derivatives. dN is the differential of the Gauss map of the surface and $(\cdot)_{T_\eta}$ means "restriction to the tangent plane T_η to the surface at $\eta = X$". Note that our proposed energy (2) can be expressed in the form of (5) if $\Phi = \left(1 - \langle I_1, I_2 \rangle / |I_1| \cdot |I_2|\right) + \alpha$, where α is just a weight for the geometric prior. In practice, we use the flow based on the first-order derivatives of Φ because it provides similar results to those of the complete expression, but saves a significant amount of computations,

$$\eta_t = \left(2H(\Phi - \Phi_N \cdot N) - \Phi_X \cdot N \right) N \,. \tag{8}$$

The level set framework has been adopted to numerically implement (8). For the technical description of the variational stereo algorithm implementation we refer to Gallego et al. (2008). We have tested the variational reconstruction algorithm using a set of images, shown in Fig. 1, acquired by Benetazzo (2006) on a water depth of 8 meters. Fig. 2 shows the successful reconstructed surface and the associated directional wave spectrum. Hereafter, we introduce the concept of the Euler characteristic (*EC*) that will be applied to predict the expected number of large maxima in oceanic sea states exploiting the high statistical content of the acquired video data via VWASS.

3 EULER CHARACTERISTIC OF RANDOM EXCURSION SETS

In algebraic topology, the **Euler characteristic** *EC* is classically defined for polyhedra according to the formula

$$EC = V - E + F \,, \tag{9}$$

where V, E, and F are respectively the numbers of vertices, edges and faces in the given polyhedron. The same definition given in (9) can be adapted to 2D surfaces which are the focus of this paper. In this case, the EC is also equivalent to the difference between the number connected components (CC) and holes (H) of the given set, viz.

$$EC = \#CC - \#H. \tag{10}$$

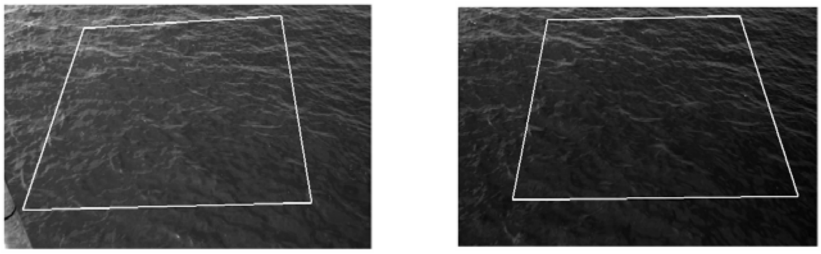

Figure 1: Input stereo pair images to the algorithm. The rectangular domain (8 m x 8.7 m) of the reconstructed surface or elevation map (right column) has been superimposed. The height of the waves is in the range ±0.2 cm.

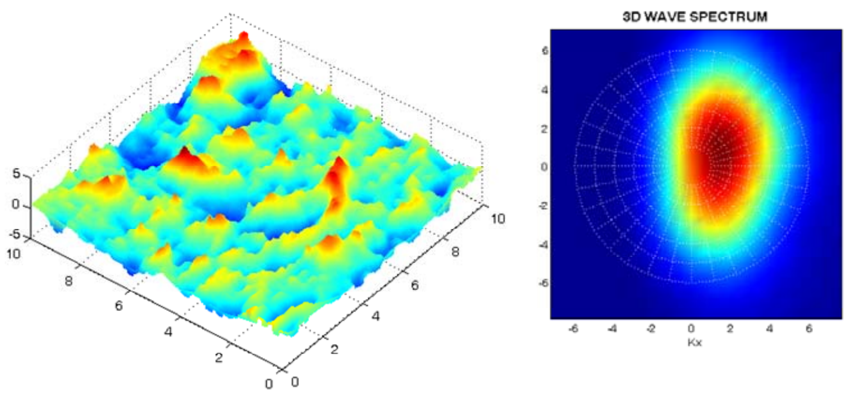

Figure 2: (left) Reconstructed normalized wave surface η via VWASS; (right) estimate directional wave spectrum of η.

For a generic 2D set Σ with complicated regions, computing the EC from the definition (10) presents some challenges. A computationally efficient approach can be devised based on (9). Following Adler (1981), we first define a Cartesian mesh grid Γ of size (Δx, Δy) that approximates the complicated domain of the given set Σ. The $EC(\Gamma)$ is then computed as follows. Denote F as the number of squares (faces) composing Γ, E_h (E_v) as the number of horizontal (vertical) segments between two neighboring mesh points and V the number of grid points. The $EC(\Gamma)$ then follows from (9) setting $E = E_h + E_v$. As the grid cell size ΔxΔy tends to zero, $EC(\Gamma) \rightarrow EC(\Sigma)$. For example, for a square EC=4-4+1=1 according to (9), which is in agreement with (10) since there is only 1 connected component and no holes.

In general, the EC of an excursion set depends very strongly on the threshold. If this is low, then EC counts the number of holes in the given set. If the threshold is high, then all the holes tend to disappear and the *EC* counts the number of connected components, or local maxima of the random field. For a stationary Gaussian field η, an exact formula for the expected value of *EC*, valid for any threshold, was discovered by Adler (1981). For 2D Gaussian fields defined over the region *S*

$$\overline{EC(A_{\eta,h})} = A_S\,(2\pi)^{-3/2}|\mathbf{\Lambda}|^{1/2}\,\xi e^{-\xi^2/2}, \qquad (11)$$

where $\overline{(\bullet)}$ means expectation, $\xi = h/\sigma$ is the normalized threshold amplitude, σ is the standard deviation of η, A_S is the area of region *S*, and $\mathbf{\Lambda}$ is the covariance matrix of the gradient $\nabla\eta$. If the excursion set touches the boundary of the area *S*, correction terms need to be added (Worsley 1995), but hereafter these will be neglected without loosing accuracy in the final results. Why the *EC* of random excursion sets is relevant to oceanic applications?

Adler (1981) and Adler & Taylor (2007) have shown that the probability that the global maximum of a random field η exceeds a threshold *h* is well approximated by the expected *EC* of the excursion set $A_{\eta,h}$, provided the threshold is high. Indeed, as the threshold *h* increases, the holes in the excursion set $A_{\eta,h}$ disappear until each of its connected components includes just one local maximum, and the *EC* counts the number of local maxima. For very large thresholds, the *EC* equals 1 if the global maximum exceeds the threshold and 0 if it is below. Thus, the $EC(A_{\eta,h})$ of large excursion sets is a binary random variable with states 0 and 1, and for *h*>>1

$$\Pr\left(\max_{P\in S}\eta(P) > h\right) = \Pr\left[EC(A_{\eta,h}) = 1\right] = \overline{EC(A_{\eta,h})}\,. \qquad (12)$$

Piterbarg (1995) also derived (12) by studying large Gaussian maxima over infinite areas. If η represents realizations of oceanic sea states at fixed time (snapshots), then the global maximum of η is the largest wave crest expected over the area *S*. Thus, (12) provides the basis for accurate estimates of exceedance probabilities of large waves by means of the *EC* of excursion sets of video images retrieved via VWASS (see Fig. 2).

A consequence of (12) is that, for *h*>>1

$$EX_{\max}(h) \approx \overline{EC(A_{\eta,h})}, \qquad (13)$$

that is, the expected number $EX_{\max}$ of large local maxima equals that of the *EC* of large excursion sets.

3.2 UPCROSSINGS AND MAXIMA OF RANDOM FIELDS

Note that for one-dimensional (1D) random processes, the *EC* of excursion sets counts the number of upcrossings. Thus, (13) simply states that the expected number of large maxima equals that of large *h*-upcrossings, implying the well known one-to-one correspondence between *h*-upcrossings and maxima at large thresholds. For two dimensional (2D) random fields this correspondence does not hold since upcrossings are contour levels. However, the definition of a 2D upcrossing is somehow vague. Can we define an appropriate 2D h-upcrossing for random fields so that the correspondence with large maxima is also one-to-one?

The answer to this question follows from the seminal work of Adler (1976) on generalizing upcrossings to higher dimensions. Without loosing generality, consider the Gaussian field η on a cartesian coordinate system (t,s) so that the covariance matrix Λ of $\nabla\eta$ is diagonal with spectral moments m_{tt} and m_{ss}, $m_{tt} > m_{ss}$, and $|\Lambda| = m_{tt}m_{ss}$. Note that the t-axis is along the principal direction θ (with respect to the original x axis) where the second spectral moment along θ attains its maximum. The partial derivatives $\partial_t\eta$ and $\partial_s\eta$ of η are thus uncorrelated and stochastically independent. With this setting in mind, a 2D h-upcrossing occurs at a point $P \in S$ if i) a 1D h-upcrossing occurs along t ($\eta = h, \partial_t\eta > 0$ at P) and ii) η attains a 1D local maximum along s, i.e. η is convex along s ($\partial_s\eta = 0,\ \partial_{ss}\eta < 0$ at P). Note that the extra condition (ii) is necessary to guaranty that, locally at P, η is a rising function. Further, this definition does not depend on the particular choice of the coordinate axes, and for large thresholds each 2D upcrossing corresponds uniquely to a large local maximum of η. Indeed, following Rice logic (Adler 1981), the expected number $EX_+(h)$ of 2D h-upcrossings is given by the following generalized Rice formula

$$EX_+(h) = A_S \int_{w_1=0}^{\infty} \int_{w_2=-\infty}^{0} w_1 |w_2| p(\eta = h, \partial_t\eta = w_1, \partial_s\eta = 0, \partial_{ss}\eta = w_2) dw_1 dw_2 \qquad (14)$$

where $p(\bullet)$ is the joint probability density function (pdf) of $\eta, \partial_t\eta, \partial_s\eta, \partial_{ss}\eta$. For an exact solution of (14) we refer to Adler (1981). Instead, an asymptotic solution for h>>1 can be derived as follows. By Gaussianity, $\partial_t\eta$ and $\partial_s\eta$ are independent of each other and from $\partial_{ss}\eta$ and η. This implies that

$$EX_+(h) = A_S \int_{w_1=0}^{\infty} w_1 p(\partial_t\eta = w_1) dw_1 \cdot \left[p(\partial_s\eta = 0) \int_{w_2=-\infty}^{0} |w_2| p(\eta = h, \partial_{ss}\eta = w_2) dw_2 \right]. \qquad (15)$$

The first integral on the left is equal to $\sqrt{m_{tt}/2\pi}$, and the term within square brackets equals the expected number, per unit length along s, of 1D local maxima with amplitude h. This is given, for large h, by $\sqrt{m_{ss}/2\pi}\,\xi \exp(-\xi^2/2)$, with $\xi = h/\sigma$ the dimensionless threshold. Noting that $|\Lambda| = m_{tt}m_{ss}$ is invariant by any axes rotation, we conclude that in general $EX_+(h)$ of (15) equals $EC(A_{\eta,h})$ of (11). Thus, for h>>1,

$$EX_+(h) \approx EX_{\max}(h) \approx EC(A_{\eta,h}). \qquad (16)$$

This proves the existence of a one-to-one correspondence between 2D upcrossings and large maxima as in 1D processes. Taylor's result (11) is thus relevant for applications because large upcrossings or maxima of random fields can be counted by simply estimating the Euler characteristic of excursion sets.

4 EULER CHARACTERISTICS OF OCEANIC SEA STATES

In the following we extend (11) to deal with the expected *EC* of excursion sets of spatial snapshots of oceanic sea states measured via VWASS (see fig. 2). To properly model oceanic nonlinearities (Fedele 2008), we follow Tayfun (1986) and define the wave surface η_{nl} over *S* as

$$\eta_{nl} = \eta + \frac{\mu}{2}\left(\eta^2 - \hat{\eta}^2\right), \tag{17}$$

where $\mu = \lambda_3/3$ is the wave steepness, which relates to the skewness λ_3 of η_{nl}, and $\hat{\eta}$ is the Hilbert transform of a normalized Gaussian field η. For ξ>>1, the excursion regions where $\eta_{nl} \geq \xi$ include just isolated local maxima. So, the structure of the excursion set can be related to the surface field locally to a maximum of η_{nl} with amplitude greater or equal to ξ. Assume that this occurs at $t = t_0$ and $s = s_0$. Then the wave surface locally around that maximum is described by the nonlinear conditional process

$$\eta_{nc} = \left\{\eta_{nl}(t,s)\middle|\eta_{nl}(t_0,s_0) \geq \xi\right\}. \tag{18}$$

From (17) it is clear that the nonlinear quadratic component of η_{nl} is phase-coupled to the extremes of the Gaussian η. So, a large maximum of η_{nl} greater or equal to ξ occurs simultaneously when η itself is at a large maximum with an amplitude greater or equal to, say ξ_1. Thus, the conditional process (18) is equivalent to the simpler process (Tayfun & Fedele 2007)

$$\eta_{nc} = \left\{\eta_{nl}(t,s)\middle|\eta(t_0,s_0) \geq \xi_1\right\} = \xi_1\Psi + \frac{\mu}{2}\xi_1^2\left(\Psi^2 - \hat{\Psi}^2\right) \qquad \mathit{for} \quad \xi_1 >> 1, \tag{19}$$

where $\Psi(t,s)$ is the normalized covariance of η. From (19), the large maximum of η_{nl} occurs at $t = t_0$ and $s = s_0$, where $\Psi = 1$ and $\hat{\Psi} = 0$, with amplitude $\xi = \xi_1 + \mu/2\ \xi_1^2$. Thus, the expected *EC* of the excursion set $\{\eta_{nl} \geq \xi\}$ equal that of the *EC* of the excursion set $\{\eta \geq \xi_1\}$ of η. By a variable transformation, from (11) it follows that

$$\overline{EC(A_{\eta_{nl},\xi})} = A_S\left(2\pi\right)^{-3/2}\left|\mathbf{\Lambda}\right|^{1/2}\frac{-1+\sqrt{(-1+2\mu\xi}}{\mu}\exp\left[-\frac{\left(-1+\sqrt{(-1+2\mu\xi}\right)^2}{2\mu^2}\right]. \tag{20}$$

Fig. 4 plots the observed *EC* and the expected *EC* (Gaussian & non-Gaussian) against the threshold *h* for the oceanic video data collected via VWASS. The data agree with the theoretical model (20).

5 CONCLUSIONS

We have proposed a novel variational image sensor (VWASS) for the stereo reconstruction of wave surfaces. The rich information content of the acquired three-dimensional video data is then exploited to compute reliable estimates the largest crest height over an area using the Euler Characteristic of random excursion sets. They provide a new statistical method for more accurate predictions of large waves during storms.

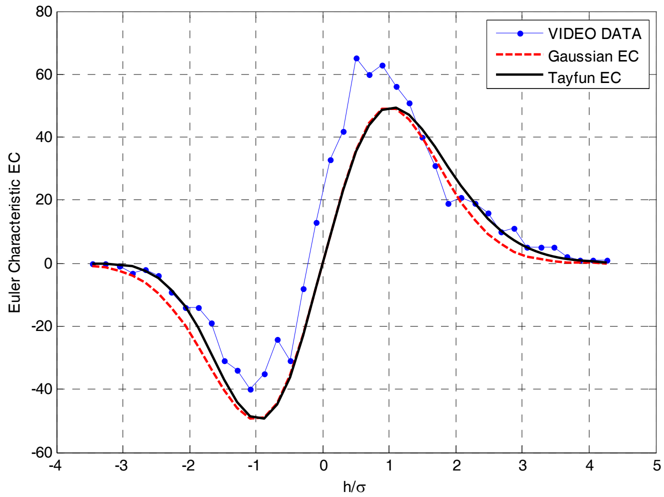

Figure 4. Observed EC and the expected EC against the threshold, as for the oceanic video data collected via VWASS shown in Fig. 2.

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