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A family of narrow-band non-linear stochastic processes for the mechanics of sea waves

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Abstract

A bi-parametric family of non-linear stochastic processes is introduced, to investigate the properties of second-order random processes with a narrow-band spectrum in the mechanics of the sea waves. In particular, the expressions of the probability density function and of the probabilities of exceedance of the absolute maximum and absolute minimum are obtained for this stochastic family. The analytical results are particularized for some processes of basic interest in the mechanics of the sea waves: the free surface displacement, and the fluctuating wave pressure beneath the sea surface.

Keywords: Mechanics of sea waves; Non-linear stochastic process; Narrow-band process; Probability density function; Probability of exceedance of the absolute maximum; Probability of exceedance of the absolute minimum

1. Introduction

The effects of non-linearity for random (wind-generated) sea waves were firstly investigated by Longuet-Higgins [1]. He achieved the first three terms of the Gram–Charlier series for the probability density function of the normalized free surface displacement, which is correct for any shape of the energy spectrum.

Later Tayfun [2] obtained the probability density function and the probability of exceedance of the crest (absolute maximum) for the free surface displacement in an undisturbed wave field. The probability of exceedance of the trough (absolute minimum) was then derived by Tung and Huang [3].

The recent book of Boccotti [4] deeply develops the linear theory of random sea waves, and the effects of finite bandwidth. As for the non-linearity effects, it is emphasized that the probability of exceedance of the absolute minimum of the fluctuating wave pressure beneath the sea surface usually is markedly greater than the probability of exceedance of the absolute maximum, especially if the waves are subject to reflection. These conclusions are based on two recent small-scale field experiments and have some important consequences in the design of submerged floating tunnels and vertical breakwaters.

In this paper a new theoretical approach is proposed to investigate the effects of non-linearity for the mechanics of the sea waves. In particular a bi-parametric family of non-linear stochastic processes is introduced, which includes the processes 'free surface displacement' and 'fluctuating wave pressure', both for waves in an undisturbed field (progressive waves) and for waves interacting with structures.

Some statistical properties of the stochastic family are derived. Firstly the characteristic function (by using the Laplace transform) and the probability density function (by inverse-Fourier transforming the characteristic function) are obtained.

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Moreover both the distributions of the absolute maximum and of the absolute minimum are achieved. All these properties depend upon two parameters α_1 and α_2 of the family.

Finally, some applications are considered: the process 'free surface displacement' and the process 'fluctuating wave pressure', both for progressive waves and for waves in front of a vertical wall. The expressions of the parameters α_1 and α_2, which enable us to quickly predict the effects of non-linearity, are obtained for the above-mentioned processes.

The new approach is valid for most of the second-order processes in the mechanics of the sea waves, except for special cases relating to the interaction of strongly non-linear waves with structures (as the fluctuating wave pressure in front of a vertical wall near the seabed on deep water).

2. Statistical properties of a stochastic family with narrow-band spectrum

Let us define the family ψ of stochastic processes, with (x, y) parameters:

$$\psi(x, y, t) = f(x, y)a\cos\big[\chi(t)\big] + g(x, y)a^2\cos^2\big[\chi(t)\big] + h(x, y)a^2\sin^2\big[\chi(t)\big], \tag{1}$$

where a is stochastic variable with Rayleigh distribution and where

$$\chi(t) = \omega_0 t + \vartheta, \tag{2}$$

where ω_0 is the angular frequency, t the time and ϑ a stochastic variable uniformly distributed in $(0, 2\pi)$.

By defining the two stochastic processes:

$$Z_1 = \frac{a\cos(\chi)}{\sigma}, \qquad Z_2 = \frac{a\sin(\chi)}{\sigma}, \tag{3}$$

where σ^2 is the variance of both the linear processes $a\cos(\chi)$ and $a\sin(\chi)$, Eq. (1) may be rewritten as:

$$\psi(Z_1, Z_2) = \sigma\big[F(x, y)Z_1 + G(x, y)Z_1^2 + H(x, y)Z_2^2\big], \tag{4}$$

where

$$\begin{aligned} &F(x, y) \equiv f(x, y), \\ &G(x, y) \equiv \sigma g(x, y), \\ &H(x, y) \equiv \sigma h(x, y). \end{aligned} \tag{5}$$

The processes (Z_1, Z_2) are both Gaussian (with zero mean value and unitary variance) and stochastically independent (Borgman [5]). Therefore the joint probability density function is given by

$$f_{Z_1,Z_2}(z_1, z_2) = \frac{1}{2\pi}\mathrm{e}^{-\frac{1}{2}(z_1^2+z_2^2)}. \tag{6}$$

From equation (4) we obtain the mean value and the variance of ψ, which are respectively given by:

$$\overline{\psi} = \sigma(G + H), \tag{7}$$

$$\sigma_\psi^2 = \frac{\sigma^2 F^2}{\beta^2}, \tag{8}$$

where

$$\beta = \frac{1}{\sqrt{1 + 2(\alpha_1^2 + \alpha_2^2)}}, \tag{9}$$

$$\alpha_1 = \frac{G}{|F|}, \qquad \alpha_2 = \frac{H}{|F|}. \tag{10}$$

Finally, we may consider the following normalized stochastic family defined as

$$\zeta = \frac{\psi - \bar{\psi}}{\sigma_\psi} = \beta\big(Z_1 + \alpha_1 Z_1^2 + \alpha_2 Z_2^2\big) - \beta(\alpha_1 + \alpha_2), \tag{11}$$

in which α_1, α_2 are deterministic parameters. The properties of the family (11) rely on these two parameters. As an example, analytical expressions of the third and fourth moments of the family ζ, are given respectively by:

$$\overline{\zeta^3} = \beta^3\big[6\alpha_1 + 8\alpha_1^3 + 8\alpha_2^3\big], \tag{12}$$

$$\overline{\zeta^4} = 3\beta^4\big(1 + 20\alpha_1^2 + 4\alpha_2^2 + 20\alpha_1^4 + 8\alpha_1^2\alpha_2^2 + 20\alpha_2^4\big). \tag{13}$$

2.1. The probability density function

Let us consider the normalized family ζ [Eq. (11)]. The characteristic function of ζ is equal to the mean value of $\mathrm{e}^{i\omega\zeta}$:

$$\overline{\mathrm{e}^{i\omega\zeta}} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{e}^{i\omega\zeta} f_{Z_1,Z_2}(z_1,z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2, \tag{14}$$

and may be rewritten as:

$$\overline{\mathrm{e}^{i\omega\zeta}} = \frac{1}{2\pi}\exp\bigl[-i\omega\beta(\alpha_1+\alpha_2)\bigr] I_1 I_2, \tag{15}$$

with the integrals I_1 and I_2 respectively defined as:

$$I_1(\omega;\alpha_1,\alpha_2) = 2\int_0^{+\infty} \cos(\omega\beta z_1)\exp\left[-\frac{z_1^2}{2}(1-2i\omega\beta\alpha_1)\right]\mathrm{d}z_1, \tag{16}$$

$$I_2(\omega;\alpha_1,\alpha_2) = 2\int_0^{+\infty} \exp\bigl(i\omega\beta\alpha_2 z_2^2\bigr)\exp\left(-\frac{1}{2}z_2^2\right)\mathrm{d}z_2. \tag{17}$$

The integrals I_1 and I_2 are evaluated by using the Laplace transform method. In particular, defining $z_1^2=t$ and $z_2^2=t$, the integrals (16) and (17) are respectively given by:

$$I_1 = \int_0^{+\infty} \exp\left[-\frac{t}{2}(1-2i\omega\beta\alpha_1)\right]\frac{\cos(\omega\beta\sqrt{t})}{\sqrt{t}}\,\mathrm{d}t = \mathbf{L}\left(\frac{\cos(\omega\beta\sqrt{t})}{\sqrt{t}}, s=\frac{1-2i\omega\beta\alpha_1}{2}\right), \tag{18}$$

$$I_2 = \int_0^{+\infty} \exp\left(-\frac{t}{2}\right)\frac{\exp(i\omega\beta\alpha_2 t)}{\sqrt{t}}\mathrm{d}t = \mathbf{L}\left(\frac{\exp(i\omega\beta\alpha_2 t)}{\sqrt{t}}, s=\frac{1}{2}\right), \tag{19}$$

where

$$\mathbf{L}\bigl[g(t),s\bigr] \equiv \int_0^{+\infty} \mathrm{e}^{-st} g(t)\,\mathrm{d}t \tag{20}$$

defines the Laplace transform of $g(t)$.

The Laplace transforms in equations (18) and (19) become, respectively:

$$\mathbf{L}\left(\frac{\cos(\lambda\sqrt{t})}{\sqrt{t}}, s\right) = \frac{\sqrt{\pi}}{\sqrt{s}}\mathrm{e}^{-\frac{\lambda^2}{4s}}, \tag{21}$$

$$\mathbf{L}\left(\frac{\mathrm{e}^{\lambda t}}{\sqrt{t}}, s\right) = \mathbf{L}\left(\frac{1}{\sqrt{t}}, s-\lambda\right) = \frac{\sqrt{\pi}}{\sqrt{s-\lambda}}, \tag{22}$$

and the characteristic function (15) is given by:

$$\overline{\mathrm{e}^{i\omega\zeta}} = \frac{\exp\bigl[-\frac{1}{2}\frac{(\omega\beta)^2}{1+4(\omega\beta\alpha_1)^2}\bigr]\exp\bigl\{-i\omega\beta\bigl[(\alpha_1+\alpha_2)+\frac{(\omega\beta)^2\alpha_1}{1+4(\omega\beta\alpha_1)^2}\bigr]\bigr\}}{\sqrt{1-4(\omega\beta)^2\alpha_1\alpha_2-2i\omega\beta(\alpha_1+\alpha_2)}}. \tag{23}$$

Finally, the probability density function f_ζ is obtained by applying the inverse Fourier transform to the characteristic function $\overline{\mathrm{e}^{i\omega\zeta}}$, that is:

$$f_\zeta(\zeta) = \mathbf{F}^{-1}\bigl(\overline{\mathrm{e}^{i\omega\zeta}},\zeta\bigr) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-i\omega\zeta}\,\overline{\mathrm{e}^{i\omega\zeta}}\,\mathrm{d}\omega, \tag{24}$$

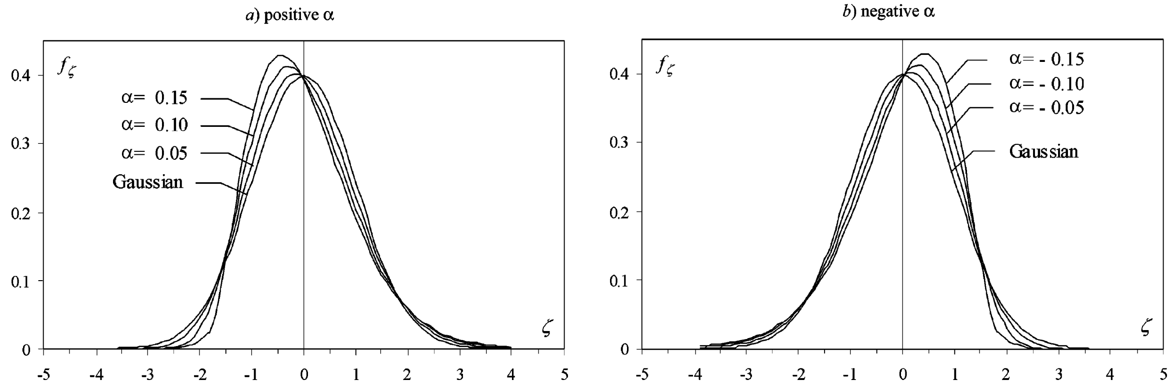

Fig. 1. The probability density function f_ζ (Eq. (30)), for fixed values of α. The f_ζ tends to the Gaussian distribution as $\alpha \to 0$.

in which $\mathbf{F}^{-1}$ is the inverse Fourier transform operator, defined as

$$\mathbf{F}^{-1}\big[f(\omega), x\big] \equiv \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{-i\omega x} f(\omega)\,\mathrm{d}\omega. \tag{25}$$

From Eqs. (23) and (24) we obtain the general expression of the probability density function f_ζ:

$$f_\zeta(\zeta) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{-i\omega\zeta} \frac{\exp\big[-\frac{1}{2}\frac{(\omega\beta)^2}{1+4(\omega\beta\alpha_1)^2}\big]\exp\big\{-i\omega\beta\big[(\alpha_1+\alpha_2)+\frac{(\omega\beta)^2\alpha_1}{1+4(\omega\beta\alpha_1)^2}\big]\big\}}{\sqrt{1-4(\omega\beta)^2\alpha_1\alpha_2 - 2i\omega\beta(\alpha_1+\alpha_2)}}\,\mathrm{d}\omega. \tag{26}$$

In the Appendix we demonstrate that Eq. (26) is real for any real ζ. Numerical integration of (26) shows also that f_ζ has positive real values for arbitrary ζ.

2.1.1. *The zero-mean value processes*

The stochastic family (1) has zero mean value if $G + H = 0$. In this case expression (4) may be rewritten as:

$$\psi = \sigma\big\lfloor F Z_1 + G\big(Z_1^2 - Z_2^2\big)\big\rfloor, \tag{27}$$

and the dimensionless process ζ (11) may be rewritten as

$$\zeta = \beta\big[Z_1 + \alpha\big(Z_1^2 - Z_2^2\big)\big], \tag{28}$$

where $\alpha = G/|F|$ and $\beta = 1/\sqrt{1+4\alpha^2}$ (let us note that $G + H = 0$ implies $\alpha_1 = -\alpha_2$ – cf. Eq. (10)). The family with zero-mean value has then only one parameter.

The expressions (12) of the third moment and (13) of the fourth moment become as the following:

$$\overline{\zeta^3} = 6\beta^3\alpha, \qquad \overline{\zeta^4} = 3\beta^4\big(1 + 24\alpha^2 + 48\alpha^4\big). \tag{29}$$

Finally, the probability density function (26) for zero-mean processes reduces itself to:

$$f_\zeta(\zeta) = \frac{1}{\pi} \int_0^{+\infty} \frac{\mathrm{e}^{-\frac{1}{2}\frac{(\omega\beta)^2}{1+4(\omega\beta\alpha)^2}}}{\sqrt{1+4(\omega\beta\alpha)^2}} \cos\bigg[\omega\bigg(\zeta + \beta\frac{(\omega\beta)^2\alpha}{1+4(\omega\beta\alpha)^2}\bigg)\bigg]\,\mathrm{d}\omega. \tag{30}$$

Fig. 1 shows the probability density function (30), for fixed values of the parameter α. Let us note that the probability density function (30) tends to the Gaussian distribution when $\alpha \to 0$ (see section 2.3).

2.2. *The distributions of the absolute maximum and of the absolute minimum*

To achieve the distribution of the absolute maximum (crest) and the distribution of the absolute minimum (trough) of the family ψ, it is convenient to rewrite Eq. (1) in the following form:

$$\psi(x, y) = f(x, y)a\cos(\chi) + \big[g(x, y) - h(x, y)\big]\frac{a^2}{2}\cos(2\chi) + \big[g(x, y) + h(x, y)\big]\frac{a^2}{2}. \tag{31}$$

The first derivative of ψ is given by

$$\frac{d\psi}{d\chi} = -a\sin(\chi)\{f(x, y) + 2[g(x, y) - h(x, y)]a\cos(\chi)\}, \tag{32}$$

and vanishes if

$$\sin(\chi) = 0 \tag{33}$$

or, for the general case of $g(x, y) \neq h(x, y)$, if

$$\cos(\chi) = -\frac{f(x, y)}{2[g(x, y) - h(x, y)]a}. \tag{34}$$

Let us suppose $f > 0$ (see Note). The values of χ which satisfy at least one of Eqs. (33) and (34) are the stationary points of the family (31). We have also that the stationary points obtained from Eq. (33) are the stationary points of the linear process $\psi_L = f(x, y)a\cos(\chi)$.

If we verify that

$$\text{the unique stationary points of } \psi \text{ are the stationary points of the linear process } \psi_{\mathrm{L}} \tag{35}$$

then the abscissa of the absolute maximum is given by $\chi_{\text{high}} = 0$ and the abscissa of the absolute minimum is given by $\chi_{\text{low}} = \pi$. Therefore from Eq. (31) we obtain the amplitudes of the absolute maximum and of the absolute minimum (in absolute value), which are given respectively by:

$$\Psi_{\text{high}} = f(x, y)a + g(x, y)a^2, \tag{36}$$

$$\Psi_{\text{low}} = f(x, y)a - g(x, y)a^2. \tag{37}$$

To achieve the probability of exceedance for the absolute maximum we define the dimensionless variable:

$$\xi_{\text{high}} = \frac{\Psi_{\text{high}}}{\sigma_\psi} = u\beta + \alpha_1\beta u^2, \tag{38}$$

where β, α_1 and α_2 are defined by Eqs. (9) and (10) and where the random variable u has Rayleigh distribution; furthermore we observe that solving equation

$$\xi = \tilde{u}\beta + \alpha_1\beta\tilde{u}^2 \tag{39}$$

with respect to the variable $\tilde{u}$, one gets the two formal roots

$$\tilde{u}_1 = -\frac{1}{2\alpha_1} - \frac{1}{2\alpha_1}\sqrt{1 + \frac{4\alpha_1\xi}{\beta}}, \qquad \tilde{u}_2 = -\frac{1}{2\alpha_1} + \frac{1}{2\alpha_1}\sqrt{1 + \frac{4\alpha_1\xi}{\beta}}. \tag{40}$$

Thus the inequality $\xi_{\text{high}} > \xi$ is verified if:

$$\xi_{\text{high}} > \xi \quad \text{if} \begin{cases} u > \tilde{u}_2 & \text{for } \alpha_1 > 0, \\ \tilde{u}_2 < u < \tilde{u}_1 & \text{for } \alpha_1 < 0, \text{ if } \xi \leqslant \beta/(4|\alpha_1|) \end{cases} \tag{41}$$

and the probability of exceedance of the absolute maximum (crest) is:

$$P(\xi_{\text{high}} > \xi) = \begin{cases} P(u > \tilde{u}_2) & \text{if } \alpha_1 > 0, \\ P(\tilde{u}_2 < u < \tilde{u}_1) & \text{if } \alpha_1 < 0 \text{ and } \xi \leqslant \beta(4|\alpha_1|), \\ 0 & \text{if } \alpha_1 < 0 \text{ and } \xi > \beta/(4|\alpha_1|), \end{cases} \tag{42}$$

where

$$P(u > \tilde{u}_2) = \exp\left[-\frac{1}{8\alpha_1^2}\left(1 - \sqrt{1 + \frac{4|\alpha_1|\xi}{\beta}}\right)^2\right], \tag{43}$$

$$P(\tilde{u}_2 < u < \tilde{u}_1) = \exp\left[-\frac{1}{8\alpha_1^2}\left(1 - \sqrt{1 - \frac{4|\alpha_1|\xi}{\beta}}\right)^2\right] - \exp\left[-\frac{1}{8\alpha_1^2}\left(1 + \sqrt{1 - \frac{4|\alpha_1|\xi}{\beta}}\right)^2\right]. \tag{44}$$

Let us note that $P(\xi_{\text{high}} > \xi)$ is the probability that an absolute maximum (Ψ_{high}) of the family ψ is greater than ξ times the standard deviation σ_ψ (see Eqs. (31), (36) and (38)).

The probability of exceedance of the absolute minimum is obtained defining the dimensionless variable

$$\xi_{\text{low}} = \frac{\Psi_{\text{low}}}{\sigma_\psi} = u\beta - \alpha_1 \beta u^2, \tag{45}$$

and is given by

$$P(\xi_{\text{low}} > \xi) = \begin{cases} P(u > \tilde{u}_2) & \text{if } \alpha_1 < 0, \\ P(\tilde{u}_2 < u < \tilde{u}_1) & \text{if } \alpha_1 > 0 \text{ and } \xi \leqslant \beta/(4|\alpha_1|), \\ 0 & \text{if } \alpha_1 > 0 \text{ and } \xi > \beta/(4|\alpha_1|). \end{cases} \tag{46}$$

Let us note that $P(\xi_{\text{low}} > \xi)$ is the probability that an absolute minimum, in absolute value (Ψ_{low}), of the family ψ is greater than ξ times the standard deviation σ_ψ (see Eqs. (31), (37) and (45)).

Finally, from Eqs. (42) and (46) we conclude that:

if $\alpha_1 > 0$:

$$\begin{aligned} &P(\xi_{\text{high}} > \xi) = P(u > \tilde{u}_2), \\ &P(\xi_{\text{low}} > \xi) = \begin{cases} P(\tilde{u}_2 < u < \tilde{u}_1) & \text{if } \xi \leqslant \beta/(4|\alpha_1|), \\ 0 & \text{if } \xi > \beta/(4|\alpha_1|), \end{cases} \end{aligned} \tag{47}$$

if $\alpha_1 < 0$:

$$\begin{aligned} &P(\xi_{\text{high}} > \xi) = \begin{cases} P(\tilde{u}_2 < u < \tilde{u}_1) & \text{if } \xi \leqslant \beta/(4|\alpha_1|), \\ 0 & \text{if } \xi > \beta/(4|\alpha_1|), \end{cases} \\ &P(\xi_{\text{low}} > \xi) = P(u > \tilde{u}_2). \end{aligned} \tag{48}$$

Fig. 2 shows the distributions of the absolute maximum and of the absolute minimum, for fixed values of parameters α_1 and for $|\alpha_2| = \alpha_1$ (let us note that the processes with $|\alpha_2| = \alpha_1$ include the zero-mean processes, for which $\alpha_1 = -\alpha_2$). Observe that for α_1 approaching zero both $P(\xi_{\text{high}} > \xi)$ and $P(\xi_{\text{low}} > \xi)$ tend to the Rayleigh distribution. For $\alpha_1 \neq 0$ the two distributions are different: in particular, if $\alpha_1 > 0$, for a fixed threshold of the probability of exceedance the absolute maximum is greater than the absolute minimum; if $\alpha_1 < 0$, for a fixed threshold of the probability of exceedance, the absolute maximum is lower than the absolute minimum.

In other words, if $f > 0$, for $\alpha_1 > 0$ (which implies $g > 0$) each realization of the process is a sequence of waves, which have crest amplitude (absolute maximum) greater than the trough amplitude (absolute minimum). For $\alpha_1 < 0$ (which implies $g < 0$) wave has the trough amplitude greater than the crest amplitude. (For the case of $f < 0$ see Note.)

Note. Let us observe that for $f < 0$ the abscissas of the absolute maximum and of the absolute minimum are, respectively $\chi_{\text{high}} = \pi$, $\chi_{\text{low}} = 0$. Furthermore $\Psi_{\text{high}} = -f(x,y)a + g(x,y)a^2$ and $\Psi_{\text{low}} = -f(x,y)a - g(x,y)a^2$; therefore if $g > 0$, we have $\alpha_1 > 0$ and the crest (in $\chi_{\text{high}} = \pi$) is greater than the trough (in $\chi_{\text{low}} = 0$); if $g < 0$ we have that $\alpha_1 < 0$ and the trough is greater than the crest. Hence, for the case $f < 0$, the expressions (47) and (48) are still valid.

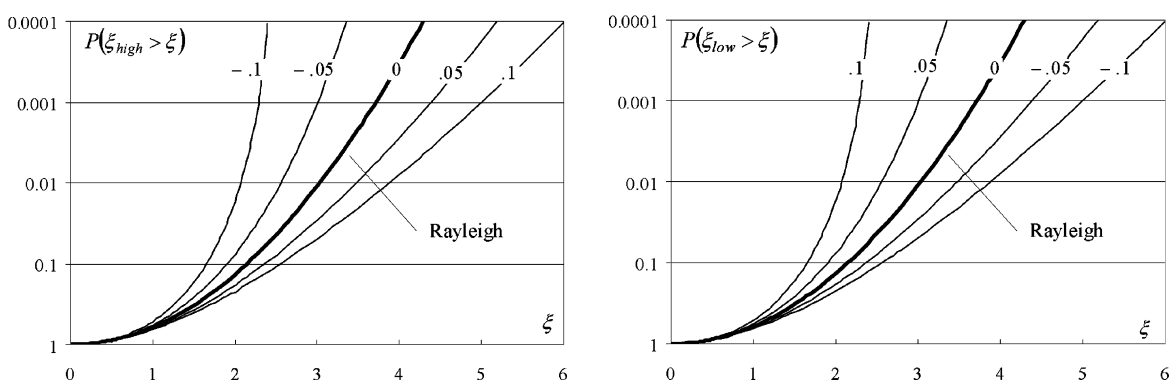

Fig. 2. The distributions of the absolute maximum $P(\xi_{\text{high}} > \xi)$ and of the absolute minimum $P(\xi_{\text{low}} > \xi)$, for fixed values of α_1 and for $|\alpha_2| = \alpha_1$.

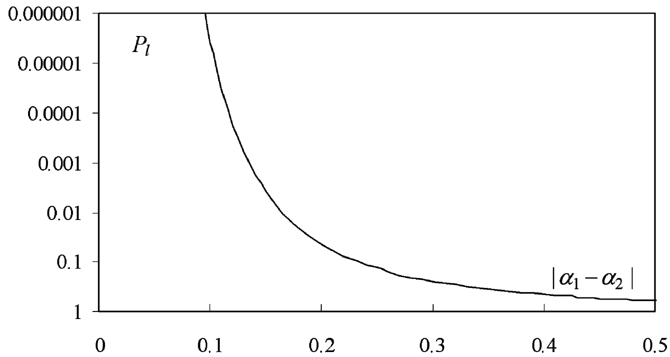

Fig. 3. The probability P_l (equation (53)) as function of $|\alpha_1 - \alpha_2|$.

2.2.1. Condition (35)

Condition (35) is always satisfied if $g(x, y) = h(x, y)$: in this case the process (31) reduces itself to the sum of the linear process ψ_L and a random constant as the following

$$\psi(x, y) = f(x, y) a \cos(\chi) + 2g(x, y)\frac{a^2}{2}. \tag{49}$$

For the general case of $g(x, y) \neq h(x, y)$, the condition (35) is verified if Eq. (34) has no solution, that is if:

$$\frac{|f(x, y)|}{|g(x, y) - h(x, y)|a/2} > 4. \tag{50}$$

In words, condition (35) is satisfied if the ratio between the amplitude of the linear component and the amplitude of the non-linear component is greater than 4 (for the sea waves this inequality is verified in most of the applications).

As function of α_1 and α_2 the inequality (50) becomes (compare to Eq. (5)):

$$\frac{1}{2|\alpha_1 - \alpha_2|} > \frac{a}{\sigma}. \tag{51}$$

a being a random variable with Rayleigh distribution, we have that

$$P\left[u > \frac{a}{\sigma}\right] = \exp\left[-\frac{1}{2}\left(\frac{a}{\sigma}\right)^2\right] \tag{52}$$

and therefore the minimum probability that the condition (35) is satisfied has expression:

$$P_l = \exp\left[-\frac{1}{8(\alpha_1 - \alpha_2)^2}\right]. \tag{53}$$

The probability P_l may be interpreted as the fraction of the realizations of the non-linear process ψ in which condition (35) is not verified. Fig. 3 shows the probability P_l as function of $|\alpha_1 - \alpha_2|$. Let observe that for $|\alpha_1 - \alpha_2| \leqslant 0.135$ the probability P_l is close to 1/1000.

2.3. Weak non-linear effects

If the parameters α_1, α_2 approach zero, the non-linear effects vanish, and each process ζ belonging to the stochastic family (11), has to converge in probability to the Gaussian process Z_1. Therefore we have that

$$\forall \varepsilon > 0 \qquad \lim_{\substack{\alpha_1 \to 0 \\ \alpha_2 \to 0}} \Pr\big(|\zeta - Z_1| \leqslant \varepsilon\big) = 1. \tag{54}$$

To verify the limit (54) we introduce the random variable

$$Y = \zeta - Z_1, \tag{55}$$

and obtain that the mean value and the variance of Y are given respectively by:

$$\overline{Y} = 0, \qquad \sigma_Y^2 = 4\left[1 - \frac{1}{\sqrt{1 + 2(\alpha_1^2 + \alpha_2^2)}}\right]. \tag{56}$$

Eq. (56) shows that the variance of Y tends to zero, if α_1 and α_2 tend to zero. Therefore the probability that the random variable Y has values equal to its mean, approaches 1:

$$\lim_{\substack{\alpha_1\to 0\\ \alpha_2\to 0}} \sigma_Y^2 = 0 \Rightarrow \lim_{\substack{\alpha_1\to 0\\ \alpha_2\to 0}} \big[\Pr(Y=0)\big] = 1, \tag{57}$$

from which we obtain

$$\lim_{\substack{\alpha_1\to 0\\ \alpha_2\to 0}} \big[\Pr(\zeta = Z_1)\big] = 1, \tag{58}$$

and thus ζ converges in probability to Z_1. The convergence in probability implies the convergence in distribution. In fact in that limit we have that $\beta \to 1$ and

$$\lim_{\substack{\alpha_1\to 0\\ \alpha_2\to 0}} f_\zeta(\zeta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-i\omega\zeta}\,\mathrm{e}^{-\omega^2/2}\,\mathrm{d}\omega = \mathbf{F}^{-1}\big(\mathrm{e}^{-\omega^2/2}, \zeta\big) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}\zeta^2}, \tag{59}$$

which is the Gaussian probability density function.

In other words, if α_1 and α_2 tend to zero, the stochastic family (11) is asymptotically Gaussian: all the processes belonging to it, tend to be Gaussian. The convergence in probability enables us to study the behaviour of the stochastic family (11) for weak non-linear effects.

By considering a small neighbourhood of $(\alpha_1 = 0, \alpha_2 = 0)$ and by retaining only the terms of order (α_1, α_2), the expressions of the third and fourth moment of ζ are given by:

$$\overline{\zeta^3} = 6\alpha_1 + \mathrm{o}(\alpha_1, \alpha_2), \tag{60}$$

$$\overline{\zeta^4} = 3 + \mathrm{o}(\alpha_1, \alpha_2). \tag{61}$$

Therefore for weak non-linear effects of order $\mathrm{O}(\alpha_1, \alpha_2)$ the process does not depend upon the value of α_2. In fact for α_1 smaller, the skewness is of order α_1, but the kurtosis is almost 3. The process is asymptotically Gaussian.

3. Applications

We consider the narrow-band processes 'free surface displacement' and 'fluctuating wave pressure', both for progressive random waves (that is waves in an undisturbed field) and for reflection of random waves (that is waves in front of a vertical wall). The reference frame (x, y) has the x-axis horizontal and the y-axis vertical, with origin on the mean water level. The bottom depth is d. The steepness ε (being $\varepsilon = k\sigma, k$ the wave number and σ the standard deviation of the linear process) of the wind-generated surface waves, in an undisturbed field, is typically between 0.05 and 0.08 (a very characteristic value is $\varepsilon = 0.055$).

3.1. The free surface displacement in an undisturbed field

The free surface displacement in an undisturbed field at any fixed point x, to the first-order solution in a Stokes expansion, is a stochastic stationary Gaussian process. The second-order free surface displacement, for narrow-band spectrum, is given by:

$$\eta(x,t) = a\cos(\chi) + ka^2 f_{\eta_1}\cos(2\chi), \tag{62}$$

where

$$f_{\eta_1}(kd) = \frac{[2+\cosh(2kd)]\cosh(kd)}{4\sinh^3(kd)}. \tag{63}$$

Assuming that the wave travels along the x-axis, we also have

$$\chi = kx - \omega_0 t + \vartheta_1, \tag{64}$$

where ϑ_1 is a stochastic variable uniformly distributed in $(0, 2\pi)$. At any fixed point x Eq. (64) is rewritten as

$$\chi = \omega_0 t + \vartheta, \tag{65}$$

where

$$\vartheta = -kx - \vartheta_1, \tag{66}$$

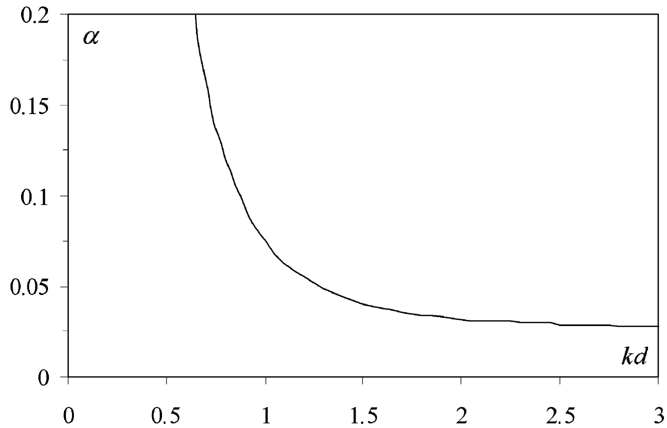

Fig. 4. The parameter α (Eq. (69)) for the free surface displacement in an undisturbed wave field as function of kd (the steepness ε has been assumed equal to 0.055).

is a stochastic variable uniformly distributed in $(0, 2\pi)$, like the ϑ_1. Therefore, the zero mean value process (62) is rewritten as the following:

$$\eta(x,t) = \sigma Z_1 + \sigma\varepsilon f_{\eta_1}\left(Z_1^2 - Z_2^2\right), \tag{67}$$

the functions Z_1 and Z_2 being defined from Eq. (3). Finally, defining

$$F = 1, \qquad G = \varepsilon f_{\eta_1}, \qquad H = -G, \tag{68}$$

we obtain that the process (67) belongs to the stochastic family (1), with parameters

$$\alpha = \alpha_1 = \varepsilon f_{\eta_1}, \qquad \alpha_2 = -\alpha_1. \tag{69}$$

Therefore the parameter α is always greater than zero. As a consequence, for a fixed threshold of probability of exceedance the wave crest (absolute maximum) is higher than the wave trough (absolute minimum). Fig. 4 shows the parameter α (obtained from Eq. (69) for $\varepsilon = 0.055$), as function of kd. On deep water ($kd \to \infty$) α tends to $\varepsilon/2$.

3.2. *The fluctuating wave pressure in an undisturbed field*

The second-order fluctuating wave pressure, for narrow-band spectrum, is given by

$$\eta_{\Delta p}(x,y,t) = a f_{\mathrm{ph}_1}\cos(\chi) + ka^2 f_{\mathrm{ph}_2}\cos(2\chi) - ka^2 f_{\mathrm{ph}_3}, \tag{70}$$

where

$$f_{\mathrm{ph}_1}(ky,kd) = \frac{\cosh[k(y+d)]}{\cosh(kd)}, \tag{71}$$

$$f_{\mathrm{ph}_2}(ky,kd) = \frac{3\cosh[2k(y+d)] - \sinh^2(kd)}{4\sinh^3(kd)\cosh(kd)}, \tag{72}$$

$$f_{\mathrm{ph}_3}(ky,kd) = \frac{\cosh[2k(y+d)] - 1}{2\sinh(2kd)}, \tag{73}$$

and where χ, assuming that the wave travels along the x-axis, is given by equation (64). As for the free surface displacement (see Section 3.1), at any fixed point (x, y) the fluctuating wave pressure (70) may be rewritten as:

$$\eta_{\Delta p}(x,y,t) = \sigma f_{\mathrm{ph}_1} Z_1 + \sigma\varepsilon(f_{\mathrm{ph}_2} - f_{\mathrm{ph}_3})Z_1^2 - \sigma\varepsilon(f_{\mathrm{ph}_2} + f_{\mathrm{ph}_3})Z_2^2. \tag{74}$$

Finally, by defining

$$F = f_{\mathrm{ph}_1}; \qquad G = \varepsilon(f_{\mathrm{ph}_2} - f_{\mathrm{ph}_3}); \qquad H = -\varepsilon(f_{\mathrm{ph}_2} + f_{\mathrm{ph}_3}); \tag{75}$$

we obtain that the process (74) belongs to family (1), with parameters

$$\alpha_1 = \varepsilon\frac{f_{\mathrm{ph}_2} - f_{\mathrm{ph}_3}}{f_{\mathrm{ph}_1}}, \qquad \alpha_2 = -\varepsilon\frac{f_{\mathrm{ph}_2} + f_{\mathrm{ph}_3}}{f_{\mathrm{ph}_1}}. \tag{76}$$

From Eq. (76) we obtain that parameter α_1 is negative for $kd > 1.32$. In this case the fluctuating wave pressure has inverse-behaviour respect to the free surface displacement: for a fixed threshold of probability of exceedance the wave crest (absolute maximum) is lower than the wave trough (absolute minimum).

The non-linear effects decrease by approaching the bottom (for a fixed kd): α_1 decreases as ky decreases.

Fig. 5 shows the family parameters (obtained from equation (76)) as function of y/d, for $\varepsilon = 0.055$.

3.3. *The free surface displacement in front of a vertical wall*

Let us consider the wave field in front of a vertical wall, located at the abscissa $x = 0$. The free surface displacement to the second-order, for an infinitely narrow spectrum, is given by

$$\eta(x,t) = 2a\cos(kx)\cos(\chi) + 2ka^2 f_{\eta 1}\cos(2kx)\cos(2\chi) + 2ka^2 f_{\eta 2}\cos(2kx) \tag{77}$$

(where χ is given by Eq. (2)) and may be rewritten as:

$$\eta(x,t) = 2\sigma\cos(kx)Z_1 + 2\sigma\varepsilon\cos(2kx)(f_{\eta 1} + f_{\eta 2})Z_1^2 + 2\sigma\varepsilon\cos(2kx)(-f_{\eta 1} + f_{\eta 2})Z_2^2, \tag{78}$$

where

$$f_{\eta 2}(kd) = \frac{1}{2\tanh(2kd)}. \tag{79}$$

This process belongs to the stochastic family (4), by defining

$$F = 2\cos(kx), \qquad G = 2\varepsilon\cos(2kx)(f_{\eta 1} + f_{\eta 2}), \qquad H = 2\varepsilon\cos(2kx)(-f_{\eta 1} + f_{\eta 2}), \tag{80}$$

and therefore it has parameters as the following expressions

$$\alpha_1 = \varepsilon(f_{\eta 1} + f_{\eta 2})\frac{\cos(2kx)}{|\cos(kx)|}, \qquad \alpha_2 = \varepsilon(-f_{\eta 1} + f_{\eta 2})\frac{\cos(2kx)}{|\cos(kx)|}. \tag{81}$$

Observe that for $kx = \pi/2 + n\pi$ (with $n = 0, \pm 1, \pm 2, \ldots$), the linear term is zero; therefore the process has only a second-order term. In this case the process does not belong to the stochastic family (1) because $\alpha_1, \alpha_2 \to \infty$.

Fig. 6 shows the family parameters α_1 and α_2 (Eq. (81)) at the wall (where $x = 0$), as function of kd. The parameter α_1 is positive. Furthermore the effects of non-linearity for surface waves on a vertical wall are greater than for surface waves in an undisturbed wave field. As an example on deep water the parameter α_2 tends to zero, and α_1 tends to ε (which is twice as much as the value of α for progressive waves on deep water).

3.4. *The fluctuating wave pressure in front of a vertical wall*

The second-order fluctuating wave pressure in front of a vertical wall, for an infinitely narrow spectrum, is given by:

$$\begin{aligned}\eta_{\Delta p}(x,y,t) = {} & 2\sigma f_{\mathrm{ph}_1}\cos(kx)Z_1 + 2\sigma\varepsilon\big[f_{\mathrm{ph}_2}\cos(2kx) - f_{\mathrm{ph}_3} + f_{\mathrm{ph}_4} \\ & + f_{\mathrm{ph}_5}\cos(2kx)\big]Z_1^2 - 2\sigma\varepsilon\big[f_{\mathrm{ph}_2}\cos(2kx) + f_{\mathrm{ph}_3} + f_{\mathrm{ph}_4} - f_{\mathrm{ph}_5}\cos(2kx)\big]Z_2^2;\end{aligned} \tag{82}$$

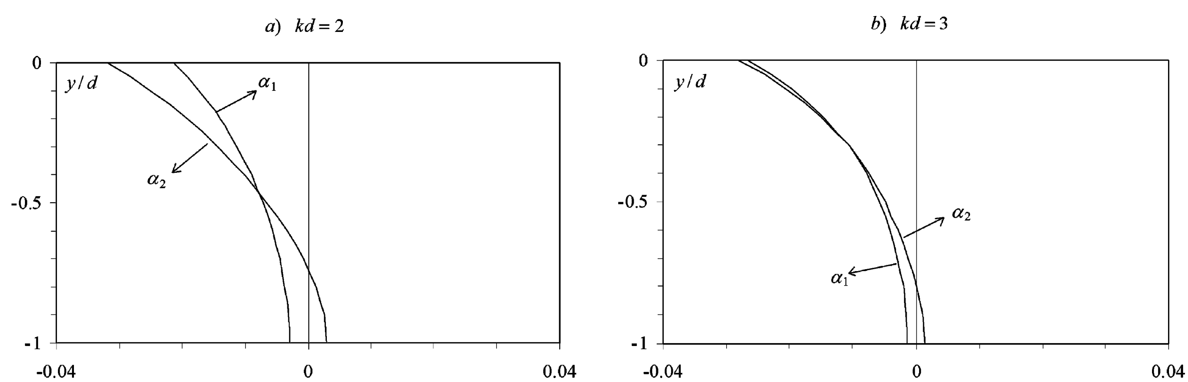

Fig. 5. The parameters α_1 and α_2 (Eq. (76)) for the fluctuating wave pressure in an undisturbed wave field, as function of y/d (the steepness ε has been assumed equal to 0.055): (a) $kd = 2$; (b) $kd = 3$.

where

$$f_{\mathrm{ph}_4}(ky,kd)=\frac{\cosh^2[k(y+d)]-\cosh(2kd)}{\sinh(2kd)}, \tag{83}$$

$$f_{\mathrm{ph}_5}(kd)=\frac{1}{2\sinh(2kd)}. \tag{84}$$

The process (82) formally belongs to the family (4) by defining

$$\begin{aligned}&F=2f_{\mathrm{ph}_1}\cos(kx);\qquad G=2\varepsilon\big[f_{\mathrm{ph}_2}\cos(2kx)-f_{\mathrm{ph}_3}+f_{\mathrm{ph}_4}+f_{\mathrm{ph}_5}\cos(2kx)\big];\\&H=-2\varepsilon\big[f_{\mathrm{ph}_2}\cos(2kx)+f_{\mathrm{ph}_3}+f_{\mathrm{ph}_4}-f_{\mathrm{ph}_5}\cos(2kx)\big].\end{aligned} \tag{85}$$

(As for the free surface displacement, the fluctuating wave pressure does not belong to the stochastic family (1), for $kx=\pi/2+n\pi$ (with $n=0,\pm1,\pm2,\ldots$).)

The parameters are given respectively by:

$$\begin{aligned}&\alpha_1=\varepsilon\frac{(f_{\mathrm{ph}_2}+f_{\mathrm{ph}_5})\cos(2kx)-f_{\mathrm{ph}_3}+f_{\mathrm{ph}_4}}{f_{\mathrm{ph}_1}|\cos(kx)|},\\&\alpha_2=-\varepsilon\frac{(f_{\mathrm{ph}_2}-f_{\mathrm{ph}_5})\cos(2kx)+f_{\mathrm{ph}_3}+f_{\mathrm{ph}_4}}{f_{\mathrm{ph}_1}|\cos(kx)|}.\end{aligned} \tag{86}$$

As an example, Fig. 7 shows the behaviour of the parameters at the wall, where $x=0$ (obtained from Eq. (86)) as function of y/d, for $\varepsilon=0.055$ and for $kd=1.5$. Being α_1 negative, for the fluctuating wave pressure each realization of the stochastic process is a sequence of waves with trough amplitude greater than the crest amplitude. Moreover, for a fixed deep kd, the

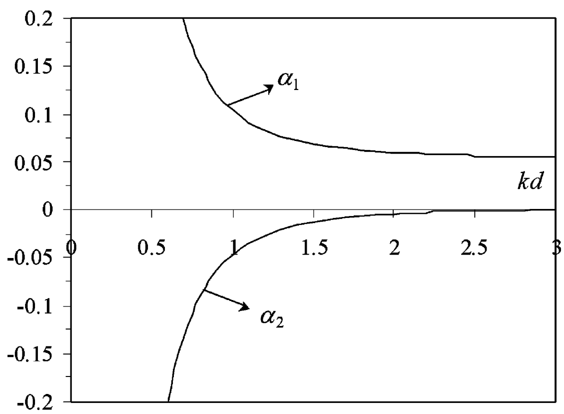

Fig. 6. The parameters α_1 and α_2 (Eq. (81)) for the free surface displacement at a vertical wall (where $x=0$), as function of kd (the steepness ε has been assumed equal to 0.055).

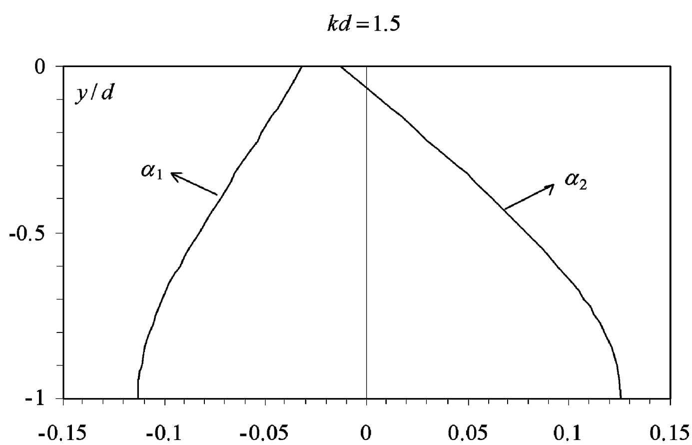

Fig. 7. The parameters α_1 and α_2 (Eq. (86)) for the fluctuating wave pressure at a vertical wall, as function of y/d. We have assumed $\varepsilon=0.055$ and $kd=1.5$.

non-linear effects increases by approaching the bottom. In fact at the bottom (where $y = -d$) for $kd \to \infty$ the fluctuating wave pressure has the limit expression as

$$\eta_{\Delta p}^{\infty} = -2\sigma\varepsilon\cos(2\chi) \tag{87}$$

in which the linear term vanishes and only the second-order term is not zero. The first linear term goes to zero faster than the second-order term, for $kd \to \infty$; this implies that $\alpha_1, \alpha_2 \to \infty$. Therefore there exists a value ky_{lim}, such that for $-kd < ky < ky_{\mathrm{lim}}$ condition (35) is not satisfied. By numerical investigation we obtain $ky_{\mathrm{lim}} \cong 1$ for characteristic value of the steepness $\varepsilon = 0.055$.

4. Conclusions

The properties of the family (1) of stochastic processes have been investigated. For this purpose the analytical expressions of the probability density function and of both the distributions of the absolute maximum and of the absolute minimum have been obtained. It is proven that all these properties depend upon two deterministic parameters named α_1 and α_2. For zero mean processes we have $\alpha_1 = -\alpha_2$, and the family has only one degree of freedom.

We have shown that if both α_1 and α_2 approach zero, the non-linearity vanishes. As a consequence the probability density function tends to be Gaussian (according to the theory of wind-generated waves of Longuet-Higgins [1] and Phillips [6]) and both the probabilities of exceedance of the absolute maximum and of the absolute minimum tend to the Rayleigh distribution (according to Longuet-Higgins [7]).

We have obtained also that for $\alpha_1 > 0$ each realization of a process belonging to the family is a sequence of waves which have the crest amplitude (absolute maximum) greater than the trough amplitude (absolute minimum, in absolute value); for $\alpha_1 < 0$ each wave has the trough amplitude greater than the crest amplitude.

Finally, in the applications we have obtained the expressions of the parameters α_1 and α_2 for some sea wave processes, as functions of ε, ky and kd. In particular we have obtained that the surface waves have the crest greater than the trough, and furthermore, for a fixed kd, the effects of non-linearity at a vertical wall are greater than the non-linear effects in an undisturbed field.

For the fluctuating wave pressure (at a point beneath the sea surface) α_1 is generally less than zero; in this case the trough of the fluctuating wave pressure are greater than the crest. Furthermore, as for the surface waves, the effects of non-linearity for the fluctuating wave pressure on a vertical wall are greater than in an undisturbed field.

These theoretical conclusions agree well with the results of two small-scale field experiments by Boccotti [4], as we can see by comparing the data and the analytical predictions for the probabilities of exceedance of the wave crest $P(\xi_{\mathrm{high}} > \xi)$ and of the wave trough $P(\xi_{\mathrm{low}} > \xi)$. Fig. 8 shows the comparison for both the free surface displacements (left panel) and the fluctuating wave pressure at a fixed depth beneath the sea surface (right panel), in an undisturbed field on deep water (we have re-examined the original source data of the small-scale field experiment described by Boccotti et al. [8]; see also Sections 10.9 and 10.10 of Boccotti [4]). Fig. 9 shows the comparison for the fluctuating wave pressure on a vertical wall (data by Boccotti [4]; see his Fig. 13.3).

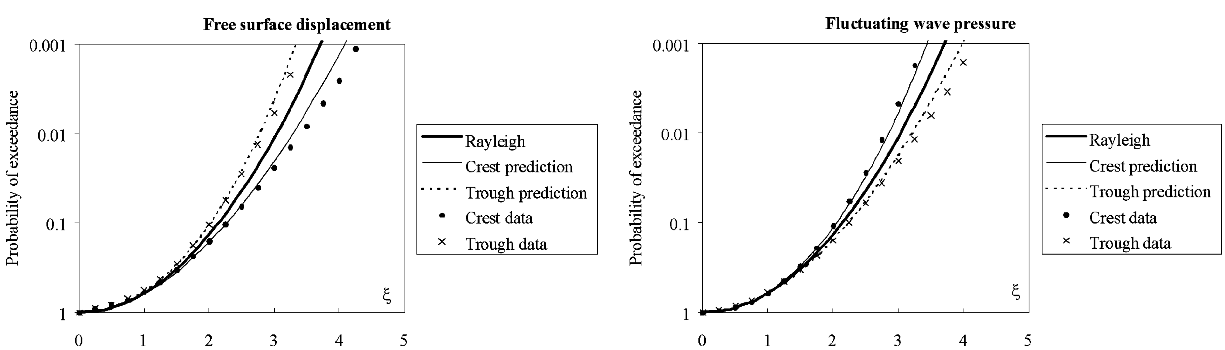

Fig. 8. The probabilities of exceedance of the wave crest $P(\xi_{\mathrm{high}} > \xi)$ and of the wave trough $P(\xi_{\mathrm{low}} > \xi)$ in an undisturbed field on deep water: comparison between the analytical predictions (for $\varepsilon = 0.055$) and the small-scale field experiment data (see Boccotti et al. [8]; Boccotti [4]). Left panel: free surface displacements. Right panel: fluctuating wave pressure at the depth $ky = -0.3$.

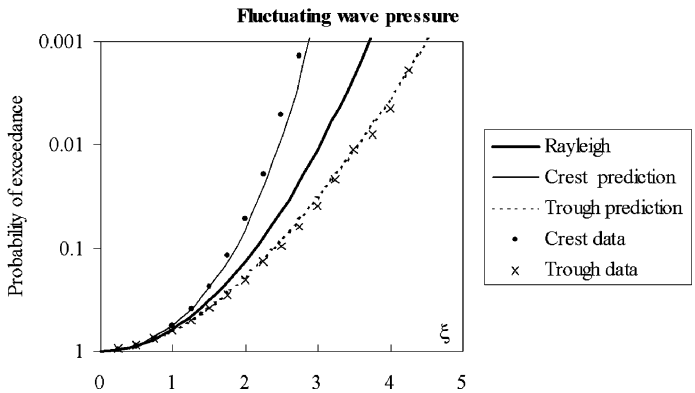

Fig. 9. The probabilities of exceedance of the wave crest $P(\xi_{\text{high}} > \xi)$ and of the wave trough $P(\xi_{\text{low}} > \xi)$ on a vertical wall ($kd = 1.26$; $y/d = -0.5$): comparison between the analytical predictions (for $\varepsilon = 0.055$) and the small-scale field experiment data (see Boccotti [4]).

Appendix

The probability density function $f_\zeta(\zeta)$ defined by Eq. (26) is real for any real ζ. To show that we take the complex conjugate of Eq. (24):

$$\left[f_\zeta(\zeta)\right]^* = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega\zeta}\left(\overline{e^{i\omega\zeta}}\right)^* d\omega \tag{88}$$

(being x^* is the complex conjugate of x); because the characteristic function $\overline{e^{i\omega\zeta}}$ satisfies the property $(\overline{e^{i\omega\zeta}})^* = \overline{e^{-i\omega\zeta}}$, Eq. (88) may be rewritten as

$$\left[f_\zeta(\zeta)\right]^* = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega\zeta}\,\overline{e^{-i\omega\zeta}}\, d\omega. \tag{89}$$

By making the change of variable $\tau = -\omega$, Eq. (89) gives

$$\left[f_\zeta(\zeta)\right]^* = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\tau\zeta}\,\overline{e^{i\tau\zeta}}\, d\tau, \tag{90}$$

from which it follows $[f_\zeta(\zeta)]^* = f_\zeta(\zeta)$ (see Eq. (24)): this implies that $f_\zeta(\zeta)$ is real.

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