A new computational paradigm for the statistics of extreme events in nonlinear random seas

Francesco Fedele*

Department of Civil Engineering, University of Vermont, Burlington, VT 05405, USA

Abstract

In this paper, the occurrence of extreme events due to the four-wave resonance interaction in weakly nonlinear water waves is investigated. The starting point is the Zakharov equation, which governs the dynamics of the spectral components of the surface displacement. It is proven that the optimal spectral components giving an extreme crest are solutions of a well-defined constrained optimization problem. A new analytical expression for the probability of exceedance of the wave crest is then proposed. The analytical results agree well with measurements data at the Draupner field and can be used for the prediction of freak wave events.

Keywords: Extreme crest; Zakharov equation; Wave-wave interaction; Energy transfer; Probability of exceedance; Freak wave

1. Introduction

Single waves that are extremely unlikely as judged by the Raleigh distribution are called freak waves. The freak event that occurred on January 1, 1995 under the Draupner platform in the North Sea [1] provides evidence that such waves can occur in the open ocean. During this freak event, an extreme crest with an amplitude of 18.5 m occurred. A mechanism which can be a cause of freak waves is related to the four-wave interaction in weakly nonlinear water waves [2]. In this context a new computational paradigm is proposed for the statistics of extreme events in nonlinear random seas.

2. The Zakharov Equation

Let us consider weakly nonlinear water waves over a finite depth d. The free-surface $\eta(\mathbf{x},t)$ is given by

$$\eta(\mathbf{x},t) = \frac{1}{2\pi}\sum \sqrt{\frac{\omega_n}{2g}}|B_n(t)| \exp[i(\mathbf{k_n} \bullet \mathbf{x} - \omega_n t + \varphi_n(t))] + c.c. \tag{1}$$

where $\varphi_n(t)$ are arbitrary time-varying phase angles, the spectral component $B_n(t)$ is defined as

$$B_n(t) = |B_n(t)| \exp[i\varphi_n(t)] \quad \forall n \tag{2}$$

and $\mathbf{x} = (x,y)$ is the horizontal spatial vector. The linearized wave frequency ω_n is related to $\mathbf{k}_n$ through the linear dispersion relation $\omega_n{}^2/g = |\mathbf{k}_n|\tanh(|\mathbf{k}_n|d)$. If third-order nonlinear effects are considered, then the spectral components $B_n(t)$ of the wave envelope satisfy the following discrete version of the Zakharov equation [3]:

$$\frac{dB_n}{dt} = -i\sum_{pqr} T_{npqr}\delta_{n+p-q-r} B_p^* B_q B_r \exp\left(i\Delta_{npqr} t\right) \tag{3}$$

Here, P^* denotes the complex conjugate of P, the kernel $T_{npqr} = T(\mathbf{k}_n,\mathbf{k}_p,\mathbf{k}_q,\mathbf{k}_r)$, $\Delta_{npqr} = \omega_n + \omega_p - \omega_q - \omega_r$ and the generalized Kronecker delta $\delta_{n+p-q-r}$ denotes that summation is taken over those subscripts satisfying $\mathbf{k}_n + \mathbf{k}_p - \mathbf{k}_q - \mathbf{k}_r = 0$.

Equation (3) admits, as motion integrals, the discrete Hamiltonian

$$H = \sum_n \omega_n B_n^* B_n + \frac{1}{2}\sum_{npqr} T_{npqr}\delta_{npqr} B_n^* B_p^* B_q B_r \exp\left(i\Delta_{npqr} t\right) \tag{4}$$

Equation (3) admits three motion integrals: the following discrete Hamiltonian, the wave action, and wave momentum, that is:

* Tel.: +1 (802) 656 8140; E-mail: ffedele@emba.uvm.edu

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$$A = \sum_n \omega_n B_n^* B_n, \mathrm{M} = \sum_n \mathrm{k}_n B_n^* B_n \quad (5)$$

3. Sufficient conditions for the occurrence of an extreme crest

The wave surface in Eq. (1) is given by the superimposition of harmonic components nonlinearly interacting among each other, according to the evolution equation (3). As time varies, a nonlinear energy transfer among the harmonic components occurs and the wave energy, action and momentum are conserved (see Eqs. (4) and (5)).

At some initial time $t = -t_0$, $B_n(t)$ is set equal to some initial conditions to be defined later, i.e.

$$B_n(t = -t_0) = \tilde{B}_n \exp(i\tilde{\varphi}_n) \quad \forall n. \quad (6)$$

In the nonlinear regime, if one imposes that at time $t = 0$ all the harmonic components are in phase, i.e.

$$\varphi_n(t = 0) = 0 \forall n \quad (7)$$

then, using Eq. (3), one can easily prove that both the spatial gradient $\nabla\eta$ and the time partial derivative $\partial\eta/\partial t$ of $\eta(\mathbf{x},t)$ vanish at ($\mathbf{x}$ = 0,t = 0). This implies that $\eta(\mathbf{x},t)$ has a stationary point at ($\mathbf{x}$ = 0,t = 0). In the following, sufficient conditions will be given such that at the stationary point ($\mathbf{x}$ = 0,t = 0) the wave surface $\eta(\mathbf{x},t)$ attains its absolute maximum, which will also be the highest crest. From Eq. (1), the wave surface amplitude at any time t at $\mathbf{x} = 0$ is

$$\eta(0,t) = \frac{1}{\pi}\sum \sqrt{\frac{\omega_n}{2g}}|B_n(t)|\cos(\omega_n t + \varphi_n(t))$$

Here, $\eta(\mathbf{0},t)$ admits the following upper bound:

$$\eta(\mathbf{0},t) \le \frac{1}{\pi}\sum \sqrt{\frac{\omega_n}{2g}}|B_n(t)| \quad (8)$$

If condition (7) is satisfied, at time $t = 0$ the surface amplitude is given by

$$H_{\max} = \frac{1}{\pi}\sum \sqrt{\frac{\omega_n}{2g}}|B_n(0)| \quad (9)$$

A sufficient condition for having an absolute maximum at ($\mathbf{x} = \mathbf{0}$,t = 0) is that $H_{\max}$ in Eq. (9) has to be greater than the right-hand side of the inequality (8), i.e.

$$\frac{1}{\pi}\sum \sqrt{\frac{\omega_n}{2g}}|B_n(0)| > \frac{1}{\pi}\sum \sqrt{\frac{\omega_n}{2g}}|B_n(t)|, \quad \forall t \quad (10)$$

Define the dimensionless variables

$$X_n = \frac{|B_n(0)|}{H}\sqrt{\frac{\omega_d}{2g}}, \tilde{X}_n = \frac{\tilde{B}_n}{H}\sqrt{\frac{\omega_d}{2g}}$$

where H is characteristic wave amplitude (hereafter the linear highest crest amplitude) and ω_d a characteristic frequency. Then, the inequality in Eq. (10) is satisfied if one can determine a set of harmonic amplitudes $|B_n(0)|$ or dimensionless variables X_n satisfying the following optimization problem:

$$\max_{X_n \in \Re^n} \frac{1}{\pi}\sum \sqrt{w_n} X_n X_n \quad X \ge 0 \quad (11)$$

subject to the constraints (4) and (5), which in terms of the X_n variables are given by

$$\sum_n w_n X_n^2 + \frac{1}{2}(\varepsilon_p \xi)^2 \sum_{npqr} \tilde{T}_{npqr} X_n X_p X_q X_r = \sum_n w_n \tilde{X}_n^2 + \frac{1}{2}(\varepsilon_p \xi)^2 \sum_{npqr} \tilde{T}_{npqr} \tilde{X}_n \tilde{X}_p \tilde{X}_q \tilde{X}_r \exp(-i\tilde{\Delta}_{npqr} t_0 + i\tilde{\Phi}_{npqr}) \quad (12)$$

and

$$\sum_n X_n^2 = \sum_n \tilde{X}_n^2, \quad \sum_n \mathrm{k}_n X_n^2 = \sum_n \mathrm{k}_n \tilde{X}_n^2 \quad (13)$$

where

$$\tilde{\Phi}_{npqr} = \tilde{\varphi}_n + \tilde{\varphi}_p - \tilde{\varphi}_q - \tilde{\varphi}_r, \quad \tilde{\Delta}_{npqr} = w_n + w_p - w_q - w_r$$

Here, $\varepsilon_p = |\mathrm{k}_d|\sigma$ is a characteristic wave steepness, $\xi = H / \sigma$ a dimensionless amplitude and σ is the standard deviation of the wave surface, $w_n = \omega_n/\omega_d$ and $\tilde{T}_{npqr} = T_{npqr}\delta_{n+p-q-r}/|\mathrm{k}_d|^3$ with $|\mathbf{k}_\mathrm{d}|$ the wave number corresponding to the characteristic frequency ω_d. For a given choice of the initial time t_0 and its relative initial conditions $\{\tilde{X}_n\}$ and $\{\tilde{\varphi}_n\}$ one can solve the optimization problem (11) and determine the optimal spectral components $\{X_n\}$. There exists a particular initial time $t = -t_0$ starting from which, the initial wave – with spectral components $\{\tilde{X}_n\}$ and phase angles $\{\tilde{\varphi}_n\}$ – nonlinearly evolves in space-time and an energy focusing occurs at ($\mathbf{x} = \mathbf{0}$,t = 0), giving the highest wave crest amplitude

$$\frac{H_{\max}}{H} = \frac{1}{\pi}\sum \sqrt{w_n} X_n. \quad (14)$$

4. The nonlinear crest amplitude and its probability of exceedance

In Gaussian seas, if an extreme crest (absolute maximum) of given elevation H occurs at ($\mathbf{x} = \mathbf{0}$,t = 0), with probability approaching 1, the surface displacement tends asymptotically to the deterministic form [4,5]

$$\eta_{\det}(\mathrm{x},t)=\frac{H}{\sigma^2}\sum_n\frac{a_n^2}{2}\exp[i(\mathbf{k}_n\bullet\mathbf{x}-\omega_n t)]+c.c. \quad (15)$$

as $\xi = H/\sigma \to \infty$. Here, the coefficients $\{a_n\}$ are related to the linear spectrum (JONSWAP spectrum in real applications)

$$E(\mathbf{k})d\mathbf{k}=\sum\frac{a_n^2}{2}\delta(\mathbf{k}-\mathbf{k}_n)d\mathrm{k}$$

At time $t = -t_0$ the initial conditions are set such that nonlinear wave surface $\eta(\mathbf{x},t)$ is equal to the linear wave group (15) and consequently the initial spectral components and phase angles are set equal to (see Eq. (6))

$$\tilde{B}_n=\pi\frac{H}{\sigma^2}\sqrt{\frac{2g}{\omega_n}}a_n^2,\quad \tilde{\varphi}_n=0$$

Thus, the linear wave group (15), which in absence of nonlinearities gives the highest crest amplitude H at $(\mathbf{x} = \mathbf{0}, t = 0)$, nonlinearly evolves according to Eq. (3) and produces a different crest amplitude $H_{\max}$ at $(\mathbf{x} = \mathbf{0}, t = 0)$. Solving the optimization problem (11) yields the relation between the highest nonlinear crest amplitude $H_{\max}$ and the linear crest amplitude H as

$$\frac{H_{\max}}{\sigma}=[1+\lambda(\xi,t_0)]\xi \quad \xi\to\infty \quad (16)$$

where the dimensionless parameter $\lambda(\xi,t_0)$ is defined as

$$\lambda(\xi,t_0)=\frac{1}{\pi}\sum\sqrt{w_n}X_n-1 \quad (17)$$

In the following, only narrow-band unidirectional waves are considered. Then, for fixed values of crest amplitude ξ, the parameter $\lambda(\xi,t_0)$ reaches a maximum $\lambda_m(\xi)$:

$$\lambda_m(\xi)=\max_{t_0\in\Re}\lambda(\xi,t_0) \quad (18)$$

at approximately $\omega_d t_0 \propto \varepsilon_d^{-2}$ which is the time scale at which the energy transfer occurs in unidirectional wave fields due to the Benjamin-Feir instability [2].

In the limit of $\xi = H/\sigma \to \infty$ the statistics of the wave crest height follows asymptotically the Rayleigh distribution

$$\Pr[H>b]=\exp\left(-\frac{b^2}{2\sigma^2}\right)$$

and the probability of exceedance of the nonlinear extreme wave crest $H_{\max}$ is given by

$$\Pr[H_{\max}>h]=\exp\left[-\frac{h^2}{2\sigma^2(1+\lambda_m(\xi))^2}\right] \quad (19)$$

If second-order effects due to bound harmonics are also taken into account, then Eq. (19) modifies as follows:

$$\Pr[H_{\max}>h]=\exp\left[-\frac{(1+\lambda_m(\xi))^2}{8\varepsilon_d^2\alpha^2}\left(1-\sqrt{1+\frac{4\varepsilon_d\alpha}{(1+\lambda_m(\xi))^2}\xi}\right)^2\right] \quad (20)$$

where one can assume $\alpha \approx 1/2$ on deep water.

5. Applications and conclusions

Consider unidirectional waves in deep water. Assume a narrow band spectrum of dimensionless bandwidth δK. Janseen [2] defines the Benjamin-Feir index (*BFI*)

$$BFI=\frac{2\sqrt{2}\varepsilon_d}{\Delta K}$$

in order to characterize the nonlinear behavior of the random field. The nonlinear energy exchange occurs in time almost periodically and produces an effect of intermittence to the surface displacement: high crests occur recurrently in time. Extreme events become more probable because of the Fermi-Pasta Ulam recurrence and the kurtosis of the wave distribution increases. Consider an initial wave spectrum with Gaussian shape

$$E(k)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{(k-1)^2}{2\sigma^2}\right]$$

Here, $k = |\mathbf{k}|/|\mathbf{k}_d|$ and the dimensionless bandwidth is assumed to be equal to the relative width at the energy level of one half of the spectrum maximum, i.e. $\Delta\mathrm{K} = 2\sigma\sqrt{2\ln 2}$.

The optimization problem (11) is solved by using the MATLAB optimization toolbox. The function $\lambda_m(\xi)$

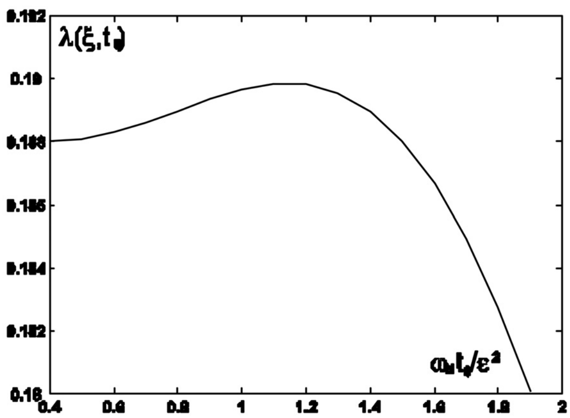

Fig. 1. The parameter $\lambda(\xi,t_0)$ for fixed value of $\xi = 2$ ($BFI = 1.15$ and $\epsilon_d = 0.05$).

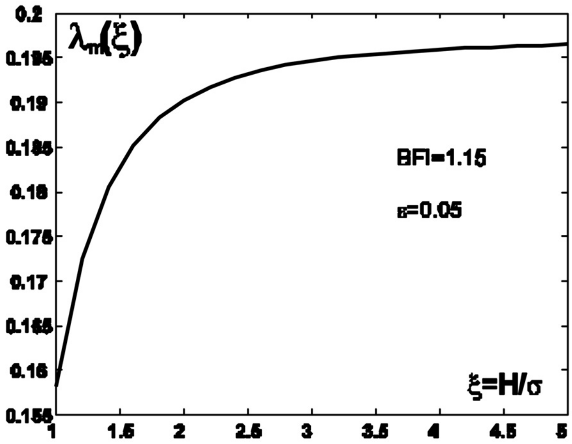

Fig. 2. The parameter $\lambda_m(\xi)$ computed for *BFI = 1.15* and ϵ_d = 0.05.

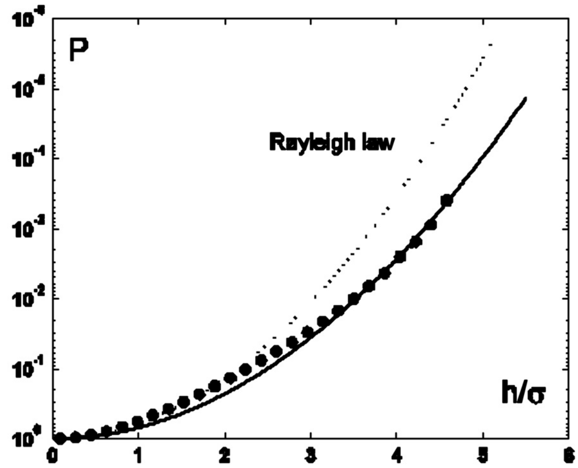

Fig. 3. Comparison between the analytical probability of exceedance of the wave crest Eq. (19) (solid line) and the empirical distribution computed from Monte-Carlo simulations (dotted line). Here, *BFI* = 1.15 and ϵ_d = 0.05.

(see Eq. (18)) is computed as follows: for fixed values of ξ, the optimization problem (11) with the constraints (12) and (13) is solved for different values of the time t_0 and the corresponding parameter $\lambda(\xi,t_0)$ is computed. As an application, consider *BFI* = 1.15 and ϵ_d = 0.05. In Fig. 1 the parameter $\lambda(\xi,t_0)$ is plotted as a function of t_0 for $\xi = 2$. As one can see from this plot, λ reaches a maximum $\lambda_m(\xi = 2) \cong 0.189$ at approximately $\omega_d t_0 \approx 1.2\epsilon$-2d in agreement with the theory. The computed

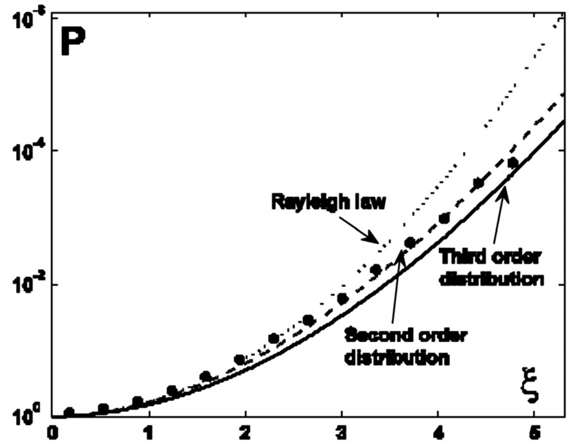

Fig. 4. Probabilities of exceedance.

function $\lambda_m(\xi)$ increases monotonically with ξ and tends to approach the asymptotic value $\lambda_{\max} = 0.195$ for $\xi \to \infty$ (see Fig. 2). Monte-Carlo simulations of the Zakharov equation are performed to validate the analytical results for the case of *BFI* = 1.15 and ϵ_d = 0.05. The empirical crest distributions agree well with the analytical distribution (19) as one can see from the plots in Fig. 3. Finally, as a real application consider the data of the wave elevation measured at the Draupner field in the central North Sea [1]. Consider a JONSWAP spectrum with *BFI = 1.1* and ϵ_d = 0.04. This choice gives the best fit with the experimental data, as one can see from Fig. 4, where the comparison between the Draupner data, the third-order distribution in Eq. (20), and the second-order distribution (Eq. (20) with $\lambda_m(\xi) = 0$) is shown.

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